

Topological and geometric properties of graph-manifolds

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Abstract

This is an exposition of results on the existence problem of π_1 -injective immersed and embedded surfaces in graph-manifolds, and also of nonpositively curved metrics on graph-manifolds, obtained by different authors. The results are represented from a unified point of view based on the notion of compatible cohomological classes and some difference equation on the graph of a graph-manifold (BKN-equation). Criteria for seven different properties of graph-manifolds at three levels are given: at the level of compatible cohomological classes; at the level of solutions to the BKN-equation; in terms of spectral properties of operator invariants of a graph-manifold.

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1 Introduction

Let M be a closed three-manifold. We say that M contains a π_1 -injectively immersed (embedded) surface, if there is an immersion (embedding) $g : S \rightarrow M$ of a closed surface S with nonpositive Euler characteristic which induces the injective homomorphism $g_* : \pi_1(S) \rightarrow \pi_1(M)$ of fundamental groups. We say that M contains a virtually and π_1 -injectively embedded surface, if some finite cover of the manifold M contains a π_1 -injectively embedded surface. We say that a manifold M is virtually fibered over the circle, if some its finite cover fibers over the circle with a fiber which is a closed surface of the negative Euler characteristic. Finally, if M carries a Riemannian metric of nonpositive sectional curvature, then we say that the manifold possesses a NPC-metric.

The properties above are significantly important for the theory of three-manifolds. One of the main conjectures of three-dimensional topology says that any closed irreducible manifold with infinite fundamental group contains a virtually and π_1 -injectively embedded surface (see [Sc]). According

a W. Thurston's conjecture any closed (and even having finite volume) hyperbolic manifold is virtually fibered over the circle (see [T, Conjecture 18]). From geometry side, the classification of closed manifolds admitting NPC-metrics is rather nontrivial (see [L], [BK2]) and by now is known up to the geometrization conjecture.

A natural problem arises, under which conditions a closed three-manifold possesses a property from the ones listed above. This survey is dedicated to solution of this problem in a class \mathfrak{M} of orientable graph-manifolds which is described below. We restrict ourself to that class by the following reasons. Firstly, listed above properties for the manifolds from \mathfrak{M} have the same nature and closely related to each other. Secondly, as the consequence of this there is a complete solution of our problem in the class \mathfrak{M} . Finally, \mathfrak{M} is a sufficiently general and wide class of closed three-manifolds.

The manifolds from \mathfrak{M} can be described as follows. A manifold which is a trivial S^1 -fibration over a surface with negative Euler characteristic and nonempty boundary is called a block. A gluing of blocks M_1 and M_2 (possibly, $M_1 = M_2$) along of boundary tori $T_1 \subset \partial M_1$ and $T_2 \subset \partial M_2$ is said to be regular if S^1 -fibers, coming up from T_1 and T_2 , are not homotopic on the gluing torus. In this case, the result of the gluing is not a block. We define \mathfrak{M}_0 to be the class of connected, closed, orientable manifolds glued in the regular way from blocks. Now, we define \mathfrak{M} as the class of orientable manifolds each of which has a finite covering by a manifold from \mathfrak{M}_0 . We also assume that every manifold $M \in \mathfrak{M}$ contains no one-sided Klein bottle (the last condition is a pure technical one and it is made for simplicity of statements).

Every manifold from \mathfrak{M} also possesses a block structure. The blocks are Seifert fibered spaces, i.e., foliations by circles which are in general not fibrations (for more detail see sect. 1.3). It is not difficult to understand that every manifold $M \in \mathfrak{M}_0$ is irreducible, i.e., every sphere (piecewise linearly) embedded in M bounds a ball. Thus all manifolds from \mathfrak{M} are also irreducible. Another invariant description of the class \mathfrak{M} is given below in sect. 1.3.2.

1.1 Properties

In fact, the solution of the characterization problem in the class \mathfrak{M} will be described for a wider spectrum of properties than indicated at the beginning of the article. Studying π_1 -injectively immersed surfaces we are interested in surfaces of negative Euler characteristic, i.e., we exclude tori (and Klein bottles) for the reason that their immersions and embeddings are easily to describe. It is well known that each π_1 -injectively immersed torus in an

irreducible graph-manifold, in particular, in a manifold $M \in \mathfrak{M}$, is homotopic to a virtually embedded one and up to an isotopy it lies in some block of M parallel to the Seifert fibration of that block (see, for example, [N3]).

Further, due to a block structure of $M \in \mathfrak{M}$, one distinguishes horizontal surfaces among surfaces immersed in such a manifold. Namely, an immersion $g : S \rightarrow M$ is said to be *horizontal*, if it is transversal in every block to the corresponding Seifert fibration. In that case, S has the negative Euler characteristic. It is known that every horizontal immersion is π_1 -injective (see [RW]).

We are interested in the following properties of a manifold $M \in \mathfrak{M}$:

- (I) M contains a π_1 -injectively immersed surface of negative Euler characteristic;
- (HI) M contains an immersed horizontal surface;
- (E) M contains a π_1 -injectively embedded surface of negative Euler characteristic;
- (VE) M contains a virtually and π_1 -injectively embedded surface of negative Euler characteristic;
- (F) M is fibered over the circle (with a surface of negative Euler characteristic as a fiber);
- (VF) M is virtually fibered over the circle;
- (NPC) M admits a NPC-metric.

It is a remarkable and not at all obvious fact that in the class \mathfrak{M} all properties listed above have actually the same nature. We explain this and give criteria for all the properties (I) – (NPC) at three levels:

- at the level of compatible cohomological classes, see Theorem 2.3;
- at the level of solutions to a difference equation on the graph of a manifold (BKN-equation), see Theorem 3.1;
- in spectral terms of operators defined via numerical invariants of a manifold, see sect. 4.

The criteria in terms of compatible cohomological classes and in terms of solutions to the BKN-equation show a common nature of the properties and allow to find various relations between them. On the other hand, these

criteria, except some rare cases, give no method to define using invariants of a manifold whether it possesses or not a property from the list (I) – (NPC). To the contrary, the criteria in spectral terms are rather constructive and allow to decide for a given manifold whether it has or not a given property. In particular, using these criteria we show that no one-directed arrow from the diagram (D) below is invertible.

The general picture of relations between the properties (I) – (NPC) can be described by the following implication diagram:

$$\begin{array}{ccccc}
 E & \Rightarrow & (VE = NPC \cup E) & \Rightarrow & I \\
 \uparrow & & \uparrow & & \Updownarrow \\
 F & \Rightarrow & VF & \Rightarrow & HI, \\
 & & \uparrow & & \\
 & & NPC & &
 \end{array} \tag{D}$$

where an implication $A \Rightarrow B$ means that if a manifold from \mathfrak{M} possesses a property A, then it possesses a property B also. All implications in this diagram, except $I \Rightarrow HI$ and $NPC \Rightarrow VF$, are almost obvious or trivial: if a manifold $M \in \mathfrak{M}$ has the property (F), then any fiber of the corresponding fibration is an embedded horizontal surface (see [WSY]); the property to be horizontal is preserved under finite coverings. On the other hand, the equality $VE = NPC \cup E$ and the implications $I \Rightarrow HI$, $NPC \Rightarrow VF$ are not trivial (see sect. 5.5.4, Corollary 5.14 and Corollary 2.4 below), as well as the fact that no one-directed arrow in the diagram (D) is invertible (see sect. 4.8).

The list of the properties above might be extended, for example, by the property of a graph-manifold to be the link of a singularity of a complex surface (see [N1], [N2]) or by the property to have a π_1 -injectively embedded surface not homological to sum of tori.

1.2 Historical remarks

Our survey is not a literature guide. Rather, we tried to give from a unified point of view as much as possible closed exposition of results obtained in this field by different authors. We use many ideas of original papers referring to them in Historical remarks scattered over the article. The survey also contains new results as well as corrects gaps and mistakes which we have found in publications. The new results are the equality $(VE)=(NPC) \cup (E)$, and the criteria for all the properties in terms of compatible cohomological classes and in terms of solutions to the BKN-equation (Theorems 2.3 and 3.1).

The properties (I) – (NPC) were studied independently by a number of authors applying different methods. Comparing the papers [BK2] and [N3]

has shown unexpected similarity in description of the properties (NPC) and (I). Analyzing this similarity we came to the conclusion that the properties have a common nature. In turn, this allowed to develop a unified approach to study of them and to discover new relations between them. Below we give a list of papers devoted to the properties (I) – (NPC) of graph-manifolds. We do not pretend that the list is complete, and apologize in advance to authors whose papers in this field missed our attention.

The property (VF). In the paper [LW], it was found a topological obstruction to this property. Namely, let M be a manifold of the class \mathfrak{M} , V be the set of its blocks (we use here notions and notations introduced in sect. 1.3.2). For each vertex $v \in V$ of the graph Γ_M of M one defines a number t_v as

$$t_v = |k_v| - \sum_{w \in \partial v} \frac{1}{|b_w|}.$$

One of the results from [LW] says if all the numbers t_v , $v \in V$ are positive, then the manifold M has no finite cover fibered over the circle. In [N2], a criterion for (VF) is proved in terms so called virtualizers. Unfortunately, that criterion allows to decide whether a given graph-manifold has the property (VF) or not only in some exceptional cases since it is unclear how to find an appropriate virtualizer. An obstacle to (VF) and a criterion similar to one from [N2] is given in [WYY]. Closely related to (VF) is the paper [RW], where a criterion for a horizontal surface to be a virtual fiber of a fibration over the circle of some finite cover of M is given. That criterion (Lemma 5.18 below) will be used in sect. 5.5. The proof of main results of [Sv1], namely the implication $\text{NPC} \Rightarrow \text{VF}$ and a spectral criterion for (VF), has a gap: the paper has no convincing argument showing the existence of rational solutions to the BKN-equation. We fill in this gap in sect. 5.3.3 and 5.8.

It is proven in [WSY] that each irreducible graph-manifold with nonempty boundary has a finite cover which fibers over the circle. This fact reflects a general principle saying that obstructions to the properties we study usually disappear for graph-manifolds with nonempty boundary.

The property (NPC). For irreducible graph-manifolds all standard obstructions to existence of NPC-metrics disappear, namely, all solvable subgroups of the fundamental group are almost abelian and centralizers virtually split (see, for example, [CE], [GW], [LY]). Thus quite unexpected was a first series of examples of graph-manifolds $M \in \mathfrak{M}_0$ without NPC-metrics, constructed in [L]. In [BK1], it is obtained a geometrization equation for a manifold $M \in \mathfrak{M}_0$, whose solvability is equivalent to existence of NPC-metrics on M . A criterion for (NPC) in terms of a quadratic form defined via numerical invariants of $M \in \mathfrak{M}_0$ is given in [BK2] (this criterion has a

little flaw, see sect. 6.6.1 at the end of the survey).

It was also shown in [L] that every irreducible graph-manifold with boundary carries an NPC-metric.

The properties (F) and (E). The obstruction from [LW] mentioned above is also an obstruction for a manifold $M \in \mathfrak{M}$ to fiber over the circle. Criteria for these properties were obtained in [N2] in terms of a reduced plumbing matrix for M . However, the criterion for (E) is inaccurate (see sect. 4.9) being rather related to the property of a graph-manifold to contain a π_1 -injectively embedded surface which is not homologous to sum of tori. A spectral criterion for (E), which is close to Theorem 4.5 below, is obtained in Thesis [Sv2].

The property (VE). Criteria for this property were obtained in [N2] in terms of virtualizers and in much more effective spectral terms close to Theorem 4.6 below.

The properties (I) and (HI). An equation on the graph of a manifold $M \in \mathfrak{M}$, whose solvability is equivalent to existence of a π_1 -injective immersion $S \rightarrow M$ of a surface of negative Euler characteristic, was obtained in [N3]. The equation turns out to be the same as the equation from [BK1], thus we call it the BKN-equation. In the same paper [N3], a criterion of solvability of the BKN-equation in spectral terms was found. The implication $I \Rightarrow HI$ is proven in [Sv2].

It was shown in [RW] that every irreducible graph-manifold with boundary contains a horizontal, properly immersed surface, the result preceding the one from [WSY] about (VF) property for graph-manifolds with boundary mentioned above.

A unified approach to study the properties (I) – (NPC) is developed in Thesis [Sv2].

1.3 Preliminaries

Here we discuss basic notions related to Seifert fibered spaces and graph-manifolds. For more detail on Seifert fibered spaces see [Sc].

1.3.1 Seifert fibered spaces

Consider the cylinder $D^2 \times [0, 1]$ fibered by the segments $\{y\} \times [0, 1]$, $y \in D^2$. Let $\varphi_{q,p} : D^2 \rightarrow D^2$ be the rotation of the disc by $2\pi q/p$, where q and p are coprime integers. The factor space $T(q, p) = D^2 \times [0, 1] / \{(y, 0) \sim (\varphi_{q,p}(y), 1)\}$ with induced fiber structure by circles is called a fibered solid torus.

Now, let M be a compact, orientable three-manifold with fiber structure by circles (we suppose for simplicity that M contains no one-sided Klein

bottle). Then M is called a Seifert fibered space, and every interior fiber $\lambda \subset M \setminus \partial M$ has a neighborhood saturated by fibers and fiber-wise homeomorphic to some fibered solid torus $T(q, p)$, where the numbers $q = q(\lambda)$ and $p = p(\lambda)$ depend in general on the fiber λ . The boundary ∂M of a Seifert fibered space consists of tori fibered by parallel circles. An interior fiber λ is said to be regular if $p(\lambda) = 1$; otherwise it is called singular. The set Λ of singular fibers is finite, all regular fibers are freely homotopic to each other, and $M \setminus \cup_{\lambda \in \Lambda} \lambda$ is a fibration by circles.

The base of a Seifert fibered space, i.e., the factor space $\mathcal{O}M$ of the fibers, is a 2-orbifold. Topologically $\mathcal{O}M$ is a surface which we denote by FM and call the underlying surface. The Euler characteristic of the 2-orbifold $\mathcal{O}M$ is defined as

$$\chi(\mathcal{O}M) = \chi(FM) - \sum_{\lambda \in \Lambda} \left(1 - \frac{1}{p(\lambda)}\right),$$

where $\chi(FM)$ is the Euler characteristic of the surface FM . If $\chi(\mathcal{O}M) < 0$, then the structure of Seifert foliation on a manifold M is unique up to isotopy (see [Sc]).

Waldhausen bases. Consider an oriented Seifert fibered space M with non-empty boundary and (orientable) base orbifold of negative Euler characteristic. Let W be the set of the boundary components. We choose an orientation of fibers in M , and let $f_w \in H_1(T_w)$ be the homological class of an oriented fiber on the boundary torus $T_w \subset \partial M$, $w \in W$. Here $H_1(T_w) = H_1(T_w; \mathbb{Q})$ is the first homology group with rational coefficients \mathbb{Q} . In what follows, we omit the notation \mathbb{Q} in (co)homology groups for simplicity.

There is an induced from M orientation on the torus T_w , hence, the intersection form $\wedge_w : H_1(T_w) \times H_1(T_w) \rightarrow \mathbb{Q}$ is well defined. Every collection of elements $\{z_w, f_w\}_{w \in W}$, where $z_w \in H_1(T_w)$, satisfying the conditions: $z = \oplus_{w \in W} z_w$ lies in the kernel of the inclusion homomorphism $i_* : H_1(\partial M) \rightarrow H_1(M)$ and $z_w \wedge_w f_w = 1$ for all $w \in W$, will be called a Waldhausen basis of the Seifert fibered space M (the elements z_w, f_w form a basis of $H_1(T_w)$ for all boundary components $w \in W$). A Waldhausen basis always exists and is defined up to the transformation $z_w \mapsto z_w + n_w f_w$, where $n_w \in \mathbb{Q}$, $\sum_{w \in W} n_w = 0$. In the case when M is a trivial S^1 -fibration over a surface F , the elements z_w can be taken from $H_1(T_w; \mathbb{Z})$ ($f_w \in H_1(T_w; \mathbb{Z})$ always), and the choice of a Waldhausen basis is equivalent to the choice of a trivialization $M = F \times S^1$.

Framed Seifert fibered spaces. Let M be a Seifert fibered space such that for every boundary torus $T_w \subset \partial M$ an element $c_w \in H_1(T_w; \mathbb{Z})$ with $c_w \wedge_w f_w \neq 0$ is fixed. Then the space M is called framed, and the collection

$C = \{c_w\}_{w \in W}$ is a framing of M . Using the following Lemma we define in sect. 1.3.2 charges, which are numerical invariants of a graph-manifold playing an important role in criteria for the properties (I) – (NPC).

Lemma 1.1. *Let $C = \{c_w\}$ be a framing of an oriented Seifert fibered space M . Then*

$$\sum_{w \in W} \frac{1}{c_w \wedge_w f_w} \cdot (i_w)_* c_w = e(M, C) \cdot f,$$

where $(i_w)_* : H_1(T_w) \rightarrow H_1(M)$ is the inclusion homomorphism induced by $i_w : T_w \hookrightarrow M$, $f \in H_1(M)$ is the homological class of a regular fiber, $(i_w)_* f_w = f$ for every $w \in W$, and $e(M, C)$ is a rational number.

Proof. Let $\{z_w, f_w\}_{w \in W}$ be a Waldhausen basis for M . Representing the class c_w as $c_w = b_w z_w + d_w f_w$, we obtain that $b_w = c_w \wedge_w f_w$ and

$$\sum_{w \in W} \frac{1}{c_w \wedge_w f_w} \cdot (i_w)_* c_w = \left(\sum_{w \in W} \frac{d_w}{b_w} \right) f.$$

The coefficient $e(M, C) = \sum_{w \in W} \frac{d_w}{b_w}$ is independent of the choice of the Waldhausen basis since the left hand side is independent of it. Hence, the claim. \square

The number $e(M, C)$ is independent neither of the framing curves orientations nor of the fiber orientation, and when the orientation of M is changed, it changes the sign. All of that immediately follows from the definitions. We shall call the number $e(M, C)$ the charge of the framed Seifert fibered space (M, C) . It differs only by sign from the relative Euler number for (M, C) introduced in [LW].

1.3.2 Graph-manifolds

At the beginning of the article, we gave a informal description of the graph-manifold classes \mathfrak{M}_0 and \mathfrak{M} . From that, it does not follow, at least immediately, that a manifold cannot have different block decompositions. Here, we give an invariant description of the classes \mathfrak{M}_0 and \mathfrak{M} , which implies uniqueness (up to isotopy) of the (maximal) block decomposition. Thus various characteristics related to the decomposition are topological invariant of a graph-manifold.

Recall that the splitting of a three-manifold M along a proper surface $\mathcal{T} \subset M$ is a manifold $M|\mathcal{T}$, which is homeomorphic to the complement $M \setminus N(\mathcal{T})$ of a regular open neighborhood $N(\mathcal{T})$ of the surface \mathcal{T} . There is a continuous projection $\pi : M|\mathcal{T} \rightarrow M$, which is a homeomorphism $\pi^{-1}(M \setminus$

$\mathcal{T}) \rightarrow M \setminus \mathcal{T}$ on the complement to \mathcal{T} and a 2-fold covering $\pi^{-1}(\mathcal{T}) \rightarrow \mathcal{T}$ over \mathcal{T} , furthermore, $\pi^{-1}(\mathcal{T}) \subset \partial(M|\mathcal{T})$.

The class \mathfrak{M} (resp. its subclass \mathfrak{M}_0) consists of 3-dimensional, connected, closed, orientable manifolds, which are not Seifert fibered spaces, having the following property. For any manifold $M \in \mathfrak{M}$ (resp. $M \in \mathfrak{M}_0$) there is a finite collection \mathcal{T} of disjoint, embedded, incompressible tori in M such that every connected component of the splitting $M|\mathcal{T}$ is a compact Seifert fibered space with orientable base orbifold of negative Euler characteristic (resp. a trivial S^1 -fibration over a surface of negative Euler characteristic). Furthermore, we assume that the manifolds from \mathfrak{M} contain no one-sided Klein bottle.

Let M be a manifold of the class \mathfrak{M} . It is well known (see [JS], [J]) that a minimal collection of tori $\mathcal{T} \subset M$ satisfying the definition is unique up to isotopy. That collection is called the JSJ-surface in M , and the connected components of the splitting $M|\mathcal{T}$ are called the maximal blocks (or the vertex manifolds) of the manifold M . Therefore, the decomposition of M into the maximal blocks and topological invariants of the maximal blocks are topological invariants of M . We note that every irreducible graph-manifold, in particular, a graph-manifold with NPC-metric, has a 2-fold cover from \mathfrak{M} and a finite cover from \mathfrak{M}_0 (see, for example, [N2], [RW]). Thus two approaches to the definition of \mathfrak{M} agree.

The graph of a graph-manifold. To every $M \in \mathfrak{M}$ one associates its graph $\Gamma = \Gamma_M$ dual to the decomposition $M = \cup_v M_v$ of the manifold into maximal blocks. In other words, the vertex set V of Γ is the set of the maximal blocks of M , and the set W of the oriented edges of Γ can be identified with the set of the boundary components of all maximal blocks. Namely, an edge $w \in W$ is directed from a vertex v to a vertex v' , if the boundary torus $T_w \subset \partial M_v$ is attached in M to the boundary torus $T_{-w} \subset \partial M_{v'}$, where the minus sign means the reverse edge orientation. The incompressible torus in M , which results from gluing of the boundary tori T_w and T_{-w} , we denote by $T_{|w|}$. The set of the edges pointing out of a vertex v is denoted by ∂v , $\partial v \subset W$, and if $w \in \partial v$, then we write $w^- = v$, $(-w)^+ = v$. The set of the nonoriented edges is denoted by E . The elements $e \in E$ are identified with the pairs $(w, -w)$, $w \in W$.

Invariants of a graph-manifold. Here we describe numerical invariants of a manifold $M \in \mathfrak{M}$ which play an important role in what follows. There are two types of them: intersection indices and charges.

We fix an orientation of M . Then every boundary torus T_w , $w \in W$, receives the induced orientation, and the orientations of T_w and T_{-w} are opposite on $T_{|w|}$. Let $a \wedge_w b \in \mathbb{Z}$ be the intersection index of integer homological

classes $a, b \in H_1(T_{|w|}; \mathbb{Z})$ w.r.t. the orientation coming up from the side of T_w . Then $a \wedge_{-w} b = -a \wedge_w b$ since the orientations of the torus $T_{|w|}$ coming up from different sides are opposite. The operation $\wedge_w : H_1(T_{|w|}; \mathbb{Z}) \times H_1(T_{|w|}; \mathbb{Z}) \rightarrow \mathbb{Z}$ is extended by linearity to an operation $\wedge_w : H_1(T_{|w|}) \times H_1(T_{|w|}) \rightarrow \mathbb{Q}$.

The intersection index. For every maximal block, we fix one of two possible orientations of its Seifert fibers. This distinguishes an element $f_w \in H_1(T_{|w|}; \mathbb{Z})$ corresponding to an oriented Seifert fiber for every boundary torus T_w . The integer

$$b_w = f_{-w} \wedge_w f_w$$

satisfies $b_w = b_{-w} \neq 0$ and it is called *the intersection index* of fibers on the torus $T_{|w|}$. It changes the sign when the manifold orientation as well as the orientation one of the fibers f_w, f_{-w} is changed.

The form of intersection indices. In general, it is not possible to find orientations in such a way that all intersection indices $b_w, w \in W$, would have one and the same sign. The obstacle to this is the cohomological class $\rho \in H^1(\Gamma; \mathbb{Z}_2)$ (described in [W, §9.6]) of the cocycle $\text{sign } b : W \rightarrow \mathbb{Z}_2$, $\text{sign } b(w) = \text{sign } b_w$ (here and in what follows we use the multiplicative form of the group \mathbb{Z}_2). We call the class $\rho = [\text{sign } b]$ *the form of intersection indices*. A 2-fold cover always exists such that its form ρ is trivial.

In what follows, we always assume that the choice of a Seifert fiber orientation for a maximal block $M_v \subset M$ means also the choice of an element $f_v \in H_1(M_v; \mathbb{Z})$ corresponding to an oriented regular fiber, and of elements $f_w \in H_1(T_{|w|}; \mathbb{Z})$ representing this fiber on the boundary tori $T_{|w|}$, $w \in \partial v$. Then $(i_w)_* f_w = f_v$, where $(i_w)_* : H_1(T_{|w|}; \mathbb{Z}) \rightarrow H_1(M_v; \mathbb{Z})$ is the inclusion homomorphism.

The charge. When fiber orientations of all maximal blocks are fixed, each of them, $M_v \subset M$, has a natural framing C_v . Namely, we take as a distinguished class $c_w \in H_1(T_{|w|}; \mathbb{Z})$, $w \in \partial v$, the class of oriented Seifert fibers of the adjacent block, $c_w = f_{-w}$. The charge

$$k_v = e(M_v, C_v) \in \mathbb{Q}$$

of the framed Seifert fibered space (M_v, C_v) (see sect. 1.3.1) is called *the charge* of the maximal block M_v . It follows from the properties of charges of framed Seifert fibered spaces that the charges $k_v, v \in V$, are independent of the choice of Seifert fiber orientations of maximal blocks, change the sign when the manifold orientation is changed, and they are topological invariant of the oriented manifold M .

The labeled graph. One associates to every oriented graph-manifold M of the class \mathfrak{M} its graph $\Gamma = \Gamma(V, W)$ together with the collection of the absolute

values of the intersection indices $|B| = \{|b_w| \in \mathbb{N} : w \in W\}$, the collection of the charges $K = \{k_v \in \mathbb{Q} : v \in V\}$ and the form of intersection indices ρ . The quadruple $(\Gamma, |B|, K, \rho)$ is called the label graph of the manifold M . A graph-manifold may not be recovered from its labeled graph. However, the information encoded in the labeled graph is sufficient to define whether a given graph-manifold possesses or not any property from the list (I) – (NPC).

In many situations the form of intersection indices ρ plays no role, then saying about the labeled graph we mean the triple $(\Gamma, |B|, K)$.

2 Compatible cohomological classes

The key to a unified approach to the properties (I) – (NPC) is the notion of a compatible collection of cohomological classes. Here we give the definition of this notion and formulate criteria for (I) – (NPC) in its terms. These criteria (Theorem 2.3) do not look much effective if we want to define whether a given manifold $M \in \mathfrak{M}$ possesses this or that property from the list (I) – (NPC). However, the criteria at the level of the BKN-equation (Theorem 3.1) and the criteria in spectral terms of operator invariants (Theorems 4.4 – 4.9), which are sufficiently effective, are based on them.

2.1 Motivation and definition

To motivate the definition of a compatible collection let us consider a π_1 -injective immersion $g : S \rightarrow M$ of a closed surface S of negative Euler characteristic in a manifold $M \in \mathfrak{M}$ (all properties (I) – (VF) imply the existence of such an immersion). Let \mathcal{T} be the JSJ-surface in M . Then g is homotopic to an immersion such that the preimage $g^{-1}(\mathcal{T})$ consists of a finite number of disjoint, simple, closed, noncontractible curves on the surface S , and the connected components of the preimage of every maximal block M_v are mapped in it either horizontally (horizontal components) or parallel to the Seifert fibers of the block (vertical annuli), see [RW]. We assume in what follows that every π_1 -injective immersion we consider has been put in this position.

Furthermore, one can assume that the vertical annuli of the immersion g , lying in one and the same block, are separated into parallel pairs. One can achieve this by taking the boundary of the collar of the surface $g(S)$. We fix an orientation of M and orientations of Seifert fibers of its maximal blocks. This induces an orientation on the horizontal components of the intersection $g(S) \cap M_v$ for every maximal block $M_v \subset M$. We orient the vertical annuli so that the parallel annuli of every pair have opposite orientations. Therefore,

a relative class $[g(S) \cap M_v] \in H_2(M_v, \partial M_v)$ of the oriented surface $g(S) \cap M_v$ is well defined. Let $l_v \in H^1(M_v)$ the class dual to $[g(S) \cap M_v]$. The collection of classes $\{l_v : v \in V\}$ depends on the choice of Seifert fiber orientations and is independent of the choice of orientation of M . This collection satisfies some conditions axiomatizing which we come to the notion of compatible cohomological classes.

For a class $l_v \in H^1(M_v)$ and an edge $w \in \partial v$ we introduce the notation $l_w := i_w^* l_v$, where the homomorphism $i_w^* : H^1(M_v) \rightarrow H^1(T_{|w|})$ is induced by the inclusion $i_w : T_{|w|} \hookrightarrow M_v$.

Lemma 2.1. *The collection of the cohomological classes $\{l_v : v \in V\}$ obtained above for a π_1 -injective immersion $g : S \rightarrow M$ satisfies the conditions: $|l_{-w}(f_w)| \leq l_w(f_w)$ and if*

$$|l_{-w}(f_w)| = l_w(f_w), \quad |l_w(f_{-w})| = l_{-w}(f_{-w}),$$

then $l_{-w} = \pm l_w$ for every edge $w \in W$ of the graph Γ_M .

Proof. Let $c \subset g(S) \cap T_{|w|}$ be the image of a simple closed curve on the surface S . Then the curve c , being a boundary component of the surface $g(S) \cap M_{w^-}$ as well as the surface $g(S) \cap M_{w^+}$, receives the orientations from both, and the orientations not necessarily coincide. The curve c is called consistent (inconsistent) if these two orientations agree (are opposite). Denote by c_w^+ (resp. c_w^-) an element of $H_1(T_{|w|})$, which is the sum of homological classes of consistent (resp. inconsistent) curves from $g(S) \cap T_{|w|}$ oriented as the boundary of the surface $g(S) \cap M_v$, where $v = w^-$ is the initial vertex of w . It follows from the definition that $c_{-w}^+ = c_w^+$ and $c_{-w}^- = -c_w^-$. Computing the value of the cocycle l_w on some $x \in H_1(T_{|w|})$, we obtain $l_w(x) = (c_w^+ + c_w^-) \wedge_w x$. On the other hand, $l_{-w}(x) = (c_{-w}^+ + c_{-w}^-) \wedge_{-w} x = -(c_w^+ - c_w^-) \wedge_w x$. Thanks to the chosen orientation of the surface $g(S) \cap M_v$, the numbers $a_w^+ = c_w^+ \wedge_w f_w$ and $a_w^- = c_w^- \wedge_w f_w$ are nonnegative. The formulae above imply $l_w(f_w) = a_w^+ + a_w^-$ and $l_{-w}(f_w) = -(a_w^+ - a_w^-)$. Therefore, for every edge $w \in W$ we have

$$|l_{-w}(f_w)| = |a_w^+ - a_w^-| \leq a_w^+ + a_w^- = l_w(f_w).$$

Suppose now that $|l_{-w}(f_w)| = l_w(f_w)$ and $|l_w(f_{-w})| = l_{-w}(f_{-w})$ for some edge $w \in W$. Then $a_w^+ \cdot a_w^- = 0 = a_{-w}^+ \cdot a_{-w}^-$. Assume that $a_w^+ = 0$. It means that either the set of the consistent curves on the torus $T_{|w|}$ is empty or the consistent curves are vertical with respect to the block M_v . In the first case, the class c_w^+ vanishes, thus $c_{-w}^+ = 0$ and $a_{-w}^+ = 0$, which easily implies the equality $l_{-w} = l_w$. In the second case, due to the choice of the orientations on parallel pairs of the vertical annuli, there are consistent as well as inconsistent curves on the torus $T_{|w|}$, which are vertical w.r.t. the block M_v , and hence

horizontal w.r.t. the adjacent block $M_{v'}$, $v' = w^+$. Thus the numbers a_{-w}^+ , a_{-w}^- are both nonzero. This is a contradiction with what precedes, and the second case is excluded. Similarly, if $a_w^- = 0$, then $l_{-w} = -l_w$. \square

An *oriented collection of cohomological classes* associates to each choice of orientations of Seifert fibers of maximal blocks a collection $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ in such a way that changing the fiber orientation of some block M_v one changes the sign of the class l_v to the opposite one. Lemma 2.1 motivates the following definition. An oriented collection of cohomological classes $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ is said to be *compatible*, if not all classes are zero and for every edge $w \in W$ the inequality $|l_{-w}(f_w)| \leq l_w(f_w)$ holds true; furthermore, if $|l_{-w}(f_w)| = l_w(f_w)$ and $|l_w(f_{-w})| = l_{-w}(f_{-w})$, then $l_{-w} = \pm l_w$.

We stress that in this definition one talks about real cohomological classes while Lemma 2.1 deals with rational ones.

By Lemma 2.1, the existence of compatible cohomological classes is a necessary condition for each property (I) – (VF). In the next section we show that this condition is also necessary for the property (NPC).

2.1.1 The case of NPC-metrics

It is well known (see, for example, the papers [GW], [LY], [E], [Sch], [B1] and the book [CE]), that every NPC-metric g on a graph-manifold M has a rather special form. Namely, JSJ-surface \mathcal{T} for M can be chosen in such a way that every torus $T_{|w|}$ is geodesic and flat, and the metric g locally splits along every maximal block M_v , i.e., every point $z \in M_v$ has a neighborhood $U_z \subset M_v$ isometric to the metric product $F_z \times (-\varepsilon, \varepsilon)$, where F_z is a surface of nonpositive Gaussian curvature. The splitting is naturally compatible with the Seifert fiber structure, the Seifert fibers are closed geodesics, and all regular fibers have one and the same length L_v .

We note that even if a block M_v is a trivial S^1 -bundle, i.e., $M_v \simeq F_v \times S^1$, the metric g may have no global splitting along M_v (see Remark 5.5 below). However, being lifted in the universal cover \widetilde{M}_v , it splits globally, and \widetilde{M}_v is isometric to the metric product $A_v \times \mathbb{R}$, where A_v is a surface of nonpositive Gaussian curvature with geodesic boundary.

As usual, we fix an orientation of M and orientations of Seifert fibers of every maximal block $M_v \subset M$. This defines an orientation of $\widetilde{M}_v = A_v \times \mathbb{R}$ and an orientation of the factor \mathbb{R} . The fundamental group $\pi_1(M_v)$ acts freely on \widetilde{M}_v by isometries leaving invariant the splitting $\widetilde{M}_v = A_v \times \mathbb{R}$. The isometries leave invariant the orientation of M since the manifold M is orientable. Furthermore, they leave invariant the orientation of the factor \mathbb{R} because M contains no one-sided Klein bottle. Therefore, we have

a homomorphism $\varphi_v : \pi_1(M_v) \rightarrow \mathbb{R}$ which assigns to every isometry $\gamma \in \pi_1(M_v)$, $\gamma : A_v \times \mathbb{R} \rightarrow A_v \times \mathbb{R}$, its shift of the factor \mathbb{R} . Since the group of the homomorphisms $\pi_1(M_v) \rightarrow \mathbb{R}$ is canonically isomorphic to $H^1(M_v; \mathbb{R})$, we get a class $l_v \in H^1(M_v; \mathbb{R})$ corresponding to the homomorphism φ_v . By the choice of orientations, $l_v(f_v) = L_v > 0$ is the length of a regular fiber.

Lemma 2.2. *The described above collection $\{l_v : v \in V\}$ of cohomological classes is compatible. Moreover, for every edge $w \in W$ we have $|l_{-w}(f_w)| < l_w(f_w)$ and $l_w(f_w) \cdot l_w(f_{-w}) = l_{-w}(f_w) \cdot l_{-w}(f_{-w})$, where, recall, $l_w = i_w^* l_v$.*

Proof. The induced flat metric on the torus $T_{|w|}$ defines a scalar product g_w on $H_1(T_{|w|}; \mathbb{R}) \simeq \mathbb{R}^2$ such that $g_w(a, a)$ is the square of the length of a closed geodesic on $T_{|w|}$ representing an element $a \in H_1(T_{|w|}; \mathbb{Z})$. Then the functional $l_w : H_1(T_{|w|}; \mathbb{R}) \rightarrow \mathbb{R}$ is the projection onto the line $\mathbb{R} \cdot f_w$ w.r.t. this scalar product, $g_w(a, f_w) = l_w(a) \cdot l_w(f_w)$ for every $a \in H_1(T_{|w|}; \mathbb{R})$, since $g_w(f_w, f_w) = L_v^2 = (l_w(f_w))^2$ is the square of the length of a regular fiber of the block M_v . The scalar products g_w and g_{-w} on the torus $T_{|w|}$ coincide, thus

$$l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = g_{-w}(f_w, f_{-w}) = g_w(f_{-w}, f_w) = l_w(f_w) \cdot l_w(f_{-w}).$$

The vectors $f_{-w}, f_w \in H_1(T_{|w|}; \mathbb{R})$ are linearly independent, thus for their scalar product we have

$$|g_{-w}(f_w, f_{-w})| < \sqrt{g_w(f_w, f_w)} \sqrt{g_w(f_{-w}, f_{-w})} = l_w(f_w) \cdot l_w(f_{-w}),$$

which implies $|l_{-w}(f_w)| < l_w(f_w)$. □

2.2 Criteria for the properties (I) – (NPC)

Now, we are able to formulate criteria for the properties (I) – (NPC) in terms of compatible cohomological classes.

Theorem 2.3. *Let M be a manifold of the class \mathfrak{M} and $\Gamma = \Gamma(V, W)$ be its graph.*

- (I) *M has the property (I) if and only if there is a compatible collection of cohomological classes for M ;*
- (HI) *M has the property (HI) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomological classes for M such that $l_v(f_v) > 0$ for every vertex $v \in V$;*

- (E) *M has the property (E) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomological classes for M such that for every edge $w \in W$ we have: if $l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0$, then $l_{-w} = \pm l_w$, where $l_w = i_w^* l_v$;*
- (VE) *M has the property (VE) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomological classes for M such that $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$;*
- (F) *M has the property (F) if and only if there are a function $\varepsilon : V \rightarrow \{\pm 1\}$ and a compatible collection $\{l_v : v \in V\}$ of cohomological classes for M such that $l_{-w}(f_w) = \varepsilon_v \varepsilon_{v'} l_w(f_w) \neq 0$ for every edge $w \in W$, where $v = w^-$, $v' = w^+$;*
- (VF) *M has the property (VF) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomological classes for M such that $l_v(f_v) > 0$ for every vertex $v \in V$ and $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$;*
- (NPC) *M has the property (NPC) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomological classes for M such that $|l_{-w}(f_w)| < l_w(f_w)$ and $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$.*

The proof of Theorem 2.3 is given in sect. 5. The implication NPC \Rightarrow VF is an immediate corollary of Theorem 2.3.

Corollary 2.4. *If a closed graph-manifold M carries a NPC-metric, then it virtually fibers over the circle.*

Proof. As we already mentioned, the manifold M has a 2-fold cover from \mathfrak{M} . Thus we can assume that $M \in \mathfrak{M}$. Comparing the criteria (VF) and (NPC) from Theorem 2.3, we obtain that M virtually fibers over the circle. \square

3 BKN-equation

By Theorem 2.3 each of the properties (I) – (NPC) is characterized by existence of corresponding compatible cohomological classes. However, Theorem 2.3 gives no method to define whether a given manifold $M \in \mathfrak{M}$ possesses or not these classes. On the other hand, it turns out that the classes considered as unknown in a sense satisfy a difference equation on the graph of the manifold which is called the BKN-equation. Its coefficients are charges and intersection indices. This allows to make one step further to the solution of the initial problem and to reformulate each of the properties (I) – (NPC) in terms of solutions to the BKN-equation.

3.1 Deriving the BKN-equation

Let M be a manifold of the class \mathfrak{M} , $\Gamma = \Gamma(V, W)$ be its graph, $X = (\Gamma, |B|, K)$ be its labeled graph. Assume that there is a compatible collection $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ of cohomological classes on M (we suppose that an orientation of M and orientations of Seifert fibers of the maximal blocks are fixed). Define functions $a : V \rightarrow \mathbb{R}$ and $\gamma : W \rightarrow \mathbb{R}$ as follows: $a(v) = l_v(f_v)$ and

$$\gamma(w) = \begin{cases} \text{sign}(b_w) l_w(f_{-w}) (l_{-w}(f_{-w}))^{-1}, & \text{if } l_{-w}(f_{-w}) \neq 0 \\ 0, & \text{if } l_{-w}(f_{-w}) = 0 \end{cases},$$

where $l_w = i_w^* l_v$, $(i_w)_* f_w = f_v \in H_1(M_v; \mathbb{Z})$ is the class representing a regular oriented fiber of M_v . We put for brevity $a_v = a(v)$, $\gamma_w = \gamma(w)$. Note that for every edge $w \in W$ the following equality holds $\gamma_w \cdot l_{-w}(f_{-w}) = \text{sign}(b_w) \cdot l_w(f_{-w})$. If $l_{-w}(f_{-w}) \neq 0$, then the equality follows from the definition of γ_w ; if $l_{-w}(f_{-w}) = 0$, then $l_w(f_{-w}) = 0$ due to compatibility of the collection $\{l_v : v \in V\}$. Thus for every vertex $v \in V$ we have

$$\begin{aligned} \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} &= \sum_{w \in \partial v} \frac{\text{sign}(b_w) l_w(f_{-w})}{|b_w|} = \sum_{w \in \partial v} \frac{i_w^* l_v(f_{-w})}{f_{-w} \wedge_w f_w} \\ &= l_v \left(\sum_{w \in \partial v} \frac{(i_w)_* f_{-w}}{f_{-w} \wedge_w f_w} \right) = l_v(k_v f_v) = k_v a_v. \end{aligned}$$

Therefore, the functions a and γ satisfy the equation

$$k_v a_v = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+}, \quad v \in V,$$

which is called the BKN-equation over the labeled graph X . The coefficients of the equation are numerical invariants of the oriented manifold M : the charges k_v , $v \in V$, and the absolute values of the intersection indices $|b_w|$, $w \in W$. The numbers γ_w , $w \in W$, as well as the charges are independent of the choice of fiber orientations of the maximal blocks, and they change the sign to the opposite one when the manifold orientation is changed.

It also follows from the compatibility condition of $\{l_v\}$ that the solution (a, γ) we have obtained has the property: $|\gamma_w| \leq 1$ and if $|\gamma_w| = |\gamma_{-w}| = 1$ for some edge $w \in W$, then $\gamma_{-w} = \gamma_w = \pm 1$.

For each of the properties (I) – (NPC) unknown functions a and γ have a clear geometric meaning. For instance, if one talks about (NPC), then $a_v = L_v$ is the length of a regular fiber of the block M_v , and

$$\gamma_w = \text{sign}(b_w) \frac{g_w(f_w, f_{-w})}{\sqrt{g_w(f_w, f_w) \cdot g_w(f_{-w}, f_{-w})}}$$

is \pm cosine of the angle between the oriented fibers f_w and f_{-w} of adjacent fibrations on the common torus $T_{|w|}$. If one talks about the property (HI), then a_v is the degree of the projection of the horizontal surface $g(S) \cap M_v$ onto the base orbifold $\mathcal{O}M_v$ of the maximal block M_v , where $g : S \rightarrow M$ is a horizontal immersion. In that case one holds $|\gamma_w| \leq 1$ by Lemma 2.1. A similar interpretation of the variables a and γ takes place for every remaining property (I), (E) – (VF). In any case $a_v \geq 0$ for all vertices $v \in V$ and $|\gamma_w| \leq 1$ for all edges $w \in W$. Admitting a conventionality, the function $a \in \mathbb{R}^V$ from a solution (a, γ) to the BKN-equation will be called the length function, and $\gamma \in \mathbb{R}^W$ will be the angle function.

3.2 Criteria for the properties (I) – (NPC)

A solution (a, γ) to the BKN-equation is said to be compatible if

- the length function a is nonnegative and $a \not\equiv 0$;
- the angle function γ satisfies the conditions: $|\gamma_w| \leq 1$ for all $w \in W$ and if $|\gamma_w \cdot \gamma_{-w}| = 1$ for some edge $w \in W$, then $\gamma_{-w} = \gamma_w = \pm 1$;
- if $a_v = 0$ for some vertex $v \in V$, then $\gamma_w = \gamma_{-w} = 0$ for all edges $w \in \partial v$.

The last condition is introduced for convenience and it is not an essential restriction because if (a, γ) is a solution to the BKN-equation, then for

$$\gamma'_w = \begin{cases} \gamma_w, & \text{if } a_{w^+} \cdot a_{w^-} \neq 0 \\ 0, & \text{if } a_{w^+} \cdot a_{w^-} = 0 \end{cases}$$

the collection (a, γ') is a solution to the BKN-equation satisfying that condition.

A solution (a, γ) to the BKN-equation is called symmetric if the angle function γ is symmetric, $\gamma_w = \gamma_{-w}$ for every edge $w \in W$. Now we are able to formulate the criteria for the properties (I) – (NPC) in terms of solutions to the BKN-equation.

Theorem 3.1. *Let $X = (\Gamma, |B|, K, \rho)$ be the labeled graph of an oriented manifold $M \in \mathfrak{M}$.*

- (I) *M has the property (I) if and only if the BKN-equation over the graph X has a compatible solution;*

- (HI) *M has the property (HI) if and only if the BKN-equation has a compatible solution (a, γ) with positive length function a , $a_v > 0$ for every vertex $v \in V$;*
- (E) *M has the property (E) if and only if the BKN-equation has a compatible, symmetric solution (a, γ) such that $\gamma_w = \gamma_{-w} = \pm 1$ for every edge $w \in W$ with $a_{w-} \cdot a_{w+} \neq 0$;*
- (VE) *M has the property (VE) if and only if the BKN-equation has a compatible, symmetric solution (a, γ) ;*
- (F) *M has the property (F) if and only if there is a compatible, symmetric solution (a, γ) to the BKN-equation with positive length function a and the angle function $\gamma : W \rightarrow \{\pm 1\}$ whose cohomological class is the form of intersection indices, $[\gamma] = \rho \in H^1(\Gamma; \mathbb{Z}_2)$;*
- (VF) *M has the property (VF) if and only if the BKN-equation has a compatible, symmetric solution (a, γ) with positive length function a ;*
- (NPC) *M has the property (NPC) if and only if the BKN-equation has a compatible, symmetric solution (a, γ) with positive length function a and $\gamma_w = \gamma_{-w} \in (-1, 1)$ for every edge $w \in W$.*

The proof of Theorem 3.1 is given in sect. 5.

3.3 Historical remarks

The BKN-equation has been obtained in [BK1], [BK2] for study the property (NPC), and independently in [N3] for study the properties (I), (HI). A relation equivalent to the BKN-equation for the particular case $|\gamma| \equiv 1$ has been obtained and used in [WSY]. This equation has been used in [BK3] to derive a criterion for the property (NPC) with finite volume for infinite graph-manifolds. In the paper [BK4], it was generalized to the case of an arbitrary dimension $n \geq 4$ for manifolds glued from maximal blocks each of which is a trivial torus bundle over a compact surface of negative Euler characteristic. In that case, in contrast to the case $n = 3$, charges are not numerical invariants, but some linear mappings. In [B2], an interpretation of the BKN-equation has been found as an analog of the Maxwell equations of classical electrodynamics. The criteria (I) – (VF) of Theorem 3.1 have been obtained in Thesis [Sv2].

4 Spectral criteria

Here we introduce operator invariants of a manifold $M \in \mathfrak{M}$, i.e., linear operators defined via the labeled graph $(\Gamma, |B|, K, \rho)$ (see sect. 1.3.2), and give criteria for the properties (I) – (NPC) in spectral terms of these operators. This completes the program of study the properties (I) – (NPC) for the manifolds of \mathfrak{M} which was indicated in Introduction.

4.1 Operator invariants

Let $\Gamma = \Gamma(V, W)$ be the graph of an oriented manifold $M \in \mathfrak{M}$, $(\Gamma, |B|, K, \rho)$ be its labeled graph. If not all charges of M are zero, then we always orient M in such a way that at least one of the charges is positive.

All operator invariants we use are linear operators on \mathbb{R}^V symmetric w.r.t. the scalar product

$$(x, x') = \sum_{v \in V} x_v x'_v, \quad x = (x_v) \in \mathbb{R}^V.$$

Thus we define them via corresponding quadratic forms on \mathbb{R}^V . Note that the space \mathbb{R}^V has a canonical basis which we identify with the vertex set V of the graph Γ .

We define symmetric operators D_M and D_M^+ via quadratic forms

$$(D_M x, x) = \sum_{v \in V} k_v x_v^2 \quad (D_M^+ x, x) = \sum_{v \in V} |k_v| x_v^2,$$

where $x = (x_v) \in \mathbb{R}^V$. For a symmetric function $\lambda : W \rightarrow \mathbb{Z}_2$ (a cocycle on the graph Γ) let us define a symmetric operator J_M^λ ,

$$(J_M^\lambda x, x) = \sum_{w \in W} \frac{\lambda_w}{|b_w|} x_{w^-} x_{w^+}$$

(recall that we always use the multiplicative form of the group \mathbb{Z}_2).

4.1.1 The operator A_M^+

We define a linear operator $A_M^+ : \mathbb{R}^V \rightarrow \mathbb{R}^V$ by

$$A_M^+ = D_M^+ - J_M,$$

where $J_M = J_M^\lambda$ for $\lambda \equiv 1$. In terms of this operator, we formulate criteria for the properties (I) and (HI). As a motivation of A_M^+ we prove the following Lemma.

Lemma 4.1. *If a manifold $M \in \mathfrak{M}$ has one of the properties from the list (I) – (NPC), then the operator A_M^+ has a nonpositive eigenvalue.*

Proof. By Theorem 3.1, the BKN-equation has a compatible solution (a, γ) . Thus

$$\begin{aligned} (A_M^+ a, a) &= \sum_{v \in V} \left(|k_v| a_v^2 - a_v \sum_{w \in \partial v} \frac{1}{|b_w|} a_{w^+} \right) \\ &\leq \sum_{v \in V} a_v \operatorname{sign}(k_v) \left(k_v a_v - \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} \right) = 0, \end{aligned}$$

because $\operatorname{sign}(k_v) \gamma_w \leq 1$. Since the operator A_M^+ is symmetric, it has a non-positive eigenvalue. \square

4.1.2 Operators A_λ

For a cocycle $\lambda : W \rightarrow \mathbb{Z}_2$, we put $A_\lambda = D_M - J_M^\lambda$. We are interested in singular properties of A_λ . Recall that a linear operator $A : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is called singular, if it has a nontrivial kernel, and supersingular, if its kernel contains an element $x \in \mathbb{R}^V$ with all coordinates x_v different from zero. We say that an operator $A : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is *weakly singular*, if there is a nonzero vector $x \in \mathbb{R}^V$ such that $(Ax)_v = 0$ for all vertices v from the support of x , $x_v \neq 0$. (The properties of supersingularity and weak singularity depend on the choice of a basis. However, the space \mathbb{R}^V has the canonical basis V , and hence these properties are manifold invariants for the corresponding operators defined below).

Lemma 4.2. *Let $\lambda : W \rightarrow \mathbb{Z}_2$ be a symmetric function. The spectrum of the operator A_λ as well as its properties to be singular, supersingular, or weakly singular depend only on the cohomological class $[\lambda] \in H^1(\Gamma; \mathbb{Z}_2)$.*

Proof. Take $\lambda' = \lambda \cdot \sigma$ where σ is a coboundary, i.e., $\sigma_w = \varepsilon_w - \varepsilon_{w^+}$ for some function $\varepsilon : V \rightarrow \mathbb{Z}_2$ and every edge $w \in W$. Given a vector $x \in \mathbb{R}^V$, we consider the vector $x' \in \mathbb{R}^V$, $x'_v = \varepsilon_v x_v$. Then for every number $\mu \in \mathbb{R}$ and every vertex $v \in V$ we have $(A_{\lambda'} x' - \mu x')_v = \varepsilon_v (A_\lambda x - \mu x)_v$, and the assertion follows. \square

We formulate spectral criteria for the properties (E) and (F) in terms of singularity of operators A_λ . By Lemma 4.2, one can assume that $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$.

4.1.3 The operator H_M

We formulate spectral criteria for the properties (VE), (VF) and (NPC) in terms of a symmetric operator $H_M : \mathbb{R}^V \rightarrow \mathbb{R}^V$ which is defined below. In that cases, one has to take into account the distribution of the charges signs on the vertex set V of the graph Γ , what essentially complicates the definition of H_M .

The graph of sign components. Let E be the set of nonoriented edges of the graph Γ . Vertices $v, v' \in V$ of Γ lie in one and the same sign component if $v = v'$ or if there is a vertex sequence $v_1 = v, \dots, v_n = v'$ such that for every $i = 1, \dots, n-1$ the vertices $v_i, v_{i+1} \in V$ are connected by an edge and $k_{v_i} k_{v_{i+1}} > 0$. This defines an equivalence relation on V . The factor set $U = V / \sim$ is called the set of sign components, and it is the disjoint union $U = U_0 \cup U_+ \cup U_-$, where $U_+(U_-)$ is the set of sign components with positive (negative) charges, and U_0 is the set of the vertices from V with zero charges. Given $u \in U$ let Γ_u be a connected subgraph in Γ spanned by the vertices from the component u , i.e., the graph Γ_u contains all edges $e \in E$ between the vertices from u . Contracting every subgraph Γ_u , $u \in U$, to a point we obtain a graph $G = G(U, E_0)$, which is called the graph of sign components of the labeled graph $(\Gamma, |B|, K)$. The vertex set of the graph G is U , and its set of (nonoriented) edges E_0 consists of all edges $e \in E$ connecting vertices from different sign components. If $U \neq U_0$, then by our assumption the set U_+ is nonempty. We denote by $p : \Gamma \rightarrow G$ the canonical projection.

Defining the function $s : U \rightarrow \{0, \pm 1\}$. Recall that a graph G is said to be bipartite if its vertex set U can be represented as the disjoint union $U = P \cup N$ in such a way that every edge of the graph connects a vertex from P with a vertex from N . This property is equivalent to that the graph G has no cycle with odd number of edges. If the graph G is connected, then the decomposition into the parts P and N is uniquely defined up to their permutation.

We define an auxiliary function $s : U \rightarrow \{0, \pm 1\}$, which will be used to define the operator H_M , as follows. If the graph $G(U, E_0)$ of the sign components of the labeled graph $(\Gamma, |B|, K)$ is not bipartite, or if $U = U_0$, then we put $s(u) = 0$ for all $u \in U$. Otherwise, we choose a decomposition $U = P \cup N$ such that $P \cap U_+ \neq \emptyset$. Now, we put $s(u) = 1$, if $u \in P$, and $s(u) = -1$, if $u \in N$.

Defining the operator H_M . For every sign component $u \in U$ we put $W_u = W \cap p^{-1}(u)$, and define symmetric operators $D_u, J_u : \mathbb{R}^u \rightarrow \mathbb{R}^u$ via the

quadratic forms

$$(D_u x, x) = s(u) \sum_{v \in u} k_v x_v^2, \quad (J_u x, x) = \sum_{w \in W_u} \frac{1}{|b_w|} x_w^- x_w^+,$$

where $x = (x_v) \in \mathbb{R}^u$ (if the set W_u is empty, then $J_u = 0$).

Now, we define a symmetric operator $H_M : \mathbb{R}^V \rightarrow \mathbb{R}^V$ by

$$H_M = \bigoplus_{u \in U} (D_u - J_u).$$

Therefore, the edges of the graph Γ connecting different sign components give no contribution into the operator H_M . If all charges of $M \in \mathfrak{M}$ are nonzero and have one and the same sign, then the graph of sign components G degenerates into a point. In that case, the operators A_M^+ , H_M coincide, $A_M^+ = H_M$.

As a motivation of H_M we prove the following Lemma.

Lemma 4.3. *If the BKN-equation over $M \in \mathfrak{M}$ has a compatible, symmetric solution (a, γ) , then $(H_M a, a) \leq 0$, in particular, the operator H_M has a nonpositive eigenvalue.*

Proof. Using the BKN-equation, we obtain

$$(H_M a, a) = \sum_{u \in U} \sum_{w \in W_u} (s(u) \gamma_w - 1) \frac{a_w^- a_w^+}{|b_w|} \leq 0.$$

The first equality in this chain of inequalities follows from that the sum of $s(u) \frac{\gamma_w}{|b_w|} a_w^- a_w^+$ over all edges $w \in W$ connecting vertices from different sign components is zero, because either $s(u) \equiv 0$, or each summand enters this sum twice with opposite signs $s(u) = 1, -1$, where $u \in U$ is the sign component from which the edge w points out. Thus the operator H_M has a nonpositive eigenvalue. \square

4.2 Spectral criterion for (I) and (HI)

Recall that a symmetric operator $A : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is said to be positive semidefinite, if $(Ax, x) \geq 0$ for every $x \in \mathbb{R}^V$. During the proof of Theorems 2.3 and 3.1 we obtain Corollary 5.14, according to which the properties (I) and (HI) are equivalent.

Theorem 4.4. *A manifold $M \in \mathfrak{M}$ has the properties (I)=(HI) if and only if one of the following conditions holds:*

- (1) *all charges of M have one and the same sign, and the operator A_M^+ is positive semidefinite and singular;*
- (2) *the operator A_M^+ has a negative eigenvalue.*

This Theorem is proved in sect. 6.1.

4.3 Spectral criterion for (E)

Theorem 4.5. *A manifold $M \in \mathfrak{M}$ has the property (E) if and only if the operator A_λ is weakly singular for some class $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$.*

This Theorem is proved in sect. 6.2.

4.4 Spectral criterion for (VE)

Theorem 4.6. *A manifold $M \in \mathfrak{M}$ has the property (VE) if and only if the operator H_M has a nonpositive eigenvalue.*

This Theorem is proved in sect. 6.3.

4.5 Spectral criterion for (F)

Theorem 4.7. *A manifold $M \in \mathfrak{M}$ has the property (F) if and only if the operator A_ρ is supersingular, where $\rho \in H^1(\Gamma; \mathbb{Z}_2)$ is the form of intersection indices.*

This Theorem is proved in sect. 6.4.

4.6 Spectral criterion for (VF)

Theorem 4.8. *A manifold $M \in \mathfrak{M}$ has the property (VF) if and only if one of the following conditions holds:*

- (1) *the operator H_M has a negative eigenvalue;*
- (2) *the operator H_M is positive semidefinite and supersingular.*

This Theorem is proved in sect. 6.5.

4.7 Spectral criterion for (NPC)

Theorem 4.9. *A manifold $M \in \mathfrak{M}$ has the property (NPC) if and only if the operator H_M has a negative eigenvalue, or the function $s : U \rightarrow \{0, \pm 1\}$ from its definition is zero.*

This Theorem is proved in sect. 6.6.

4.8 The implication diagram

Using the spectral criteria (Theorems 4.4 – 4.9), we give here examples, which show that no one-directed arrow of the diagram (D) is invertible, and also that there are manifolds of the class \mathfrak{M} possessing no property from the list (I) – (NPC).

4.8.1 Graph-manifolds $M(\alpha)$

All manifolds from the examples below belong to the class \mathfrak{M}_0 and they all have one and the same graph Γ , which is a linear graph with three vertices v_1, v_2, v_3 and two (nonoriented) edges e_1, e_2 , connecting the vertices v_1, v_2 and v_2, v_3 respectively. The maximal blocks $M_i = M_{v_i}$, $i = 1, 2, 3$ have the following structure. The blocks M_1 and M_3 are trivial bundles over the torus with hole, and the block M_2 is the trivial bundle over the torus with two holes. The gluing between the blocks M_1 and M_2 is one and the same for all examples, and gluings between M_2 and M_3 are parameterized by matrices $\alpha \in GL(2, \mathbb{Z})$ with determinant equals -1 .

To describe these gluings, we fix trivializations $M_i = F_i \times S^1$ of the blocks, orientations of S^1 -factors of all blocks and orientations of the surfaces F_i , $i = 1, 2, 3$. For every boundary torus T_w , $w \in W$, this fixes a basis z_w, f_w of the group $H_1(M_v; \mathbb{Z})$, where $w \in \partial v$. Every element z_w represents the corresponding oriented boundary component of F_v , and f_w represents the oriented factor S^1 of the block M_v . We suppose that $e_1 = (w_1, -w_1)$, $e_2 = (w_2, -w_2)$, where $w_1 \in \partial v_1$, $-w_1, w_2 \in \partial v_2$ and $-w_2 \in \partial v_3$. Then the gluing $g : T_{-w_1} \rightarrow T_{w_1}$ between the blocks M_1, M_2 is given by

$$\begin{aligned} g(z_{-w_1}) &= f_{w_1} \\ g(f_{-w_1}) &= z_{w_1} + f_{w_1}. \end{aligned}$$

For

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Z})$$

with $ad - bc = -1$ the gluing $h_\alpha : T_{-w_2} \rightarrow T_{w_2}$ between the blocks M_2 and M_3 is given by α , i.e.,

$$\begin{aligned} h_\alpha(z_{-w_2}) &= az_{w_2} + cf_{w_2} \\ h_\alpha(f_{-w_2}) &= bz_{w_2} + df_{w_2}. \end{aligned}$$

We denote by $M(\alpha)$ the graph-manifold resulting from the gluings.

The orientations of the S^1 -factors of its maximal blocks and the base surfaces define orientations of the blocks themselves which are compatible under the gluings because the last reverse the orientation of the corresponding boundary tori. This induces an orientation of the manifold $M(\alpha)$. Its intersection indices are

$$b_1 := b_{w_1} = b_{-w_1} = 1, \quad b_2 := b_{w_2} = b_{-w_2} = b,$$

and the charges can be easily found using Lemma 1.1:

$$k_1 := k_{v_1} = 1, \quad k_2 := k_{v_2} = \frac{d}{b}, \quad k_3 := k_{v_3} = -\frac{a}{b}.$$

4.8.2 A graph-manifold without (I) – (NPC)

An example of such a manifold is $M(\alpha) \in \mathfrak{M}_0$ with the gluing

$$\alpha = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}.$$

The operator $A_\alpha^+ = A_{M(\alpha)}^+$ is given by the matrix

$$A_\alpha^+ = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

whose eigenvalues $1, 2 \pm \sqrt{3}$, are positive. By Lemma 4.1, the manifold $M(\alpha)$ possesses no property from the list (I) – (NPC).

4.8.3 (I) \nRightarrow (VE)

Here we give an example $M(\alpha)$ with property (I) and without (VE). This also shows that (HI) \nRightarrow (VF). As α we take the matrix

$$\alpha = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

The operator $A_\alpha^+ = A_{M(\alpha)}^+$ is given by the matrix

$$A_\alpha^+ = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and it has a negative eigenvalue, $1 - \sqrt{2}$. By Theorem 4.4, the manifold $M(\alpha)$ has the property (I). On the other hand, the operator $H_{M(\alpha)}$ is the identity, $H_{M(\alpha)} = \text{id}$. Thus by Theorem 4.6, the manifold $M(\alpha)$ does not have the property (VE).

4.8.4 (VE) $\not\Rightarrow$ (E)

Here we give an example $M(\alpha)$ with property (VE) and without (E). As α we take the matrix

$$\alpha = \begin{bmatrix} -3 & 2 \\ -1 & 1 \end{bmatrix}.$$

The operator $H_\alpha = H_{M(\alpha)}$ is given by the matrix

$$H_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1/2 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix}$$

and it has a negative eigenvalue because $\det H_\alpha = -1$. By Theorem 4.6, the manifold $M(\alpha)$ has the property (VE). The graph Γ of the manifold is simply connected, thus the group $H^1(\Gamma; \mathbb{Z}_2)$ is trivial, and the operator A_λ coincides with H_α for every $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$. The principal minors of the matrix H_α of every order are different from zero, thus the operator $A_\lambda = H_\alpha$ is not weakly singular. By Theorem 4.5, the manifold $M(\alpha)$ does not have the property (E).

4.8.5 (VE) $\not\Rightarrow$ (VF)

Here we give an example $M(\alpha)$ with property (VE) and without (VF). As α we take the matrix

$$\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

The operator $H_\alpha = H_{M(\alpha)}$ is given by the matrix

$$H_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and it is singular. By Theorem 4.6, the manifold $M(\alpha)$ has the property (VE). On the other hand, the operator H_α is positive semidefinite and not supersingular because $x_1 = x_2 = 0$ for every vector $x = (x_1, x_2, x_3)$ with $H_\alpha x = 0$. By Theorem 4.8, the manifold $M(\alpha)$ does not have the property (VF).

4.8.6 (E) \nRightarrow (F), (VF) \nRightarrow (F), (VF) \nRightarrow (NPC)

Here we give an example $M(\alpha)$ with properties (E), (VF) and without (F), (NPC). As α we take the matrix

$$\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since the group $H^1(\Gamma; \mathbb{Z}_2)$ is trivial, the operator A_λ is given by the matrix

$$A_\lambda = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

for every $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$, and it is weakly singular because $A_\lambda x = (0, 0, -1)$ for $x = (1, 1, 0)$. By Theorem 4.5, the manifold $M(\alpha)$ has the property (E).

The operator $H_\alpha = H_{M(\alpha)}$ is given by the matrix

$$H_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and thus it is positive semidefinite and supersingular, $H_\alpha x = 0$ for $x = (1, 1, 1)$. By Theorem 4.8, the manifold $M(\alpha)$ virtually fibers over the circle. On the other hand, the operator $A_\rho = A_\lambda$ is not singular, $\det A_\lambda = -1$. By Theorem 4.7, the manifold $M(\alpha)$ does not fibers over the circle. The operator H_α has no negative eigenvalue, and the function s from its definition is nonzero. By Theorem 4.9, the manifold $M(\alpha)$ carries no NPC-metric.

4.9 Historical remarks

A spectral criterion for the property (F) equivalent to Theorem 4.7, and a criterion for the property (VE) close to Theorem 4.6 were obtained in [N2]. A spectral criterion for the property (I) equivalent to Theorem 4.4 (but without the equality (I)=(HI)) was obtained in [N3]. In the same paper, an example of a graph-manifold from the class \mathfrak{M} without the property (I) has been

found, and it was shown that (I) $\not\Rightarrow$ (VE). A spectral criterion for the property (E) was obtained in [Sv2], and for the property (VF) was formulated in [Sv1]. A spectral criterion for the property (NPC) was obtained in [BK2] (see, however, sect. 6.6.1). The example 4.8.6 is a counterexample to Theorem D.1(3) from [N2] since the manifold $M(\alpha)$ from that example has the property (E), and at the same time the matrix A_λ (“decomposition matrix” in terms of [N2]) is not singular.

5 Proof of Theorems 2.3 and 3.1

We prove Theorems 2.3 and 3.1 simultaneously. The first step of the proof is the following simple but important Lemma about extension.

Lemma 5.1. *Assume that for a block M_v of an oriented manifold $M \in \mathfrak{M}$ an orientation of Seifert fibers is fixed and a Waldhausen basis $\{z_w, f_w\}_{w \in \partial v}$ is chosen. Given a collection of cohomological classes $\{l_w \in H^1(T_{|w|}; \mathbb{R}) : w \in \partial v\}$, there is a class $l_v \in H^1(M_v; \mathbb{R})$ such that $i_w^* l_v = l_w$ for all $w \in \partial v$ if and only if the following conditions hold*

- (1) $l_w(f_w) = a_v$ is independent of $w \in \partial v$;
- (2) $\sum_{w \in \partial v} l_w(z_w) = 0$.

Proof. The conditions (1) and (2) are necessary since for every class $l_v \in H^1(M_v; \mathbb{R})$ with $i_w^* l_v = l_w$, $w \in \partial v$, we have: the number $l_w(f_w) = l_v(f_v)$ is independent of $w \in \partial v$ and $\sum_{w \in \partial v} l_w(z_w) = l_v(\oplus_{w \in \partial v} (i_w)_* z_w) = 0$.

Conversely, assume that a collection of cohomological classes $\{l_w\}$ satisfies the conditions (1) and (2). We show that the class $m_v = \bigoplus_{w \in \partial v} l_w \in H^1(\partial M_v; \mathbb{R})$ lies in the image of the homomorphism $i^* : H^1(M_v; \mathbb{R}) \rightarrow H^1(\partial M_v; \mathbb{R})$. To this end, it suffices to show, due to the exact cohomological sequence of the pair $(M_v, \partial M_v)$, that $m_v(\partial S) = 0$ for every relative class $S \in H_2(M_v, \partial M_v; \mathbb{R})$, where $\partial : H_2(M_v, \partial M_v; \mathbb{R}) \rightarrow H_1(\partial M_v; \mathbb{R})$ is the boundary homomorphism. Representing the class ∂S as $\partial S = \bigoplus_{w \in \partial v} c_w$ and decomposing the element c_w over the basis z_w, f_w , we obtain $c_w = \alpha z_w + \beta_w f_w$, where the coefficient α is independent of $w \in \partial v$, because the classes ∂S and $z = \bigoplus_{w \in \partial v} z_w$ from $H_1(\partial M_v; \mathbb{R})$ lie in the kernel of the homomorphism i_* . Thus

$$m_v(\partial S) = \sum_{w \in \partial v} l_w(c_w) = \left(\sum_{w \in \partial v} \beta_w \right) \cdot a_v.$$

On the other hand,

$$0 = i_*(\partial S) = \sum_{w \in \partial v} (i_w)_*(c_w) = \left(\sum_{w \in \partial v} \beta_w \right) \cdot f_v.$$

Consequently, $m_v(\partial S) = 0$ and the class m_v lies in the image of the homomorphism i^* . Then any class $l_v \in H^1(M_v; \mathbb{R})$ with $i^*l_v = m_v$ is a required one. \square

5.1 Theorems 2.3 and 3.1 are equivalent

As a corollary of Lemma 5.1 we obtain that Theorems 2.3 and 3.1 are equivalent in the following sense.

Proposition 5.2. *Let $M \in \mathfrak{M}$ an oriented manifold. For each property (A) from the list (I) – (NPC) we have: a compatible collection of cohomological classes on M satisfying the condition (A) of Theorem 2.3 exists if and only if the BKN-equation over M has a compatible solution satisfying the condition (A) of Theorem 3.1.*

Proof. Let $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ be compatible cohomological classes satisfying the condition (A) of Theorem 2.3 (we assume that fiber orientations of the maximal blocks are fixed). Then the functions $a : V \rightarrow \mathbb{R}$, $a(v) = l_v(f_v)$, and $\gamma : W \rightarrow \mathbb{R}$,

$$\gamma(w) = \begin{cases} \text{sign}(b_w)l_w(f_{-w})(l_{-w}(f_{-w}))^{-1}, & \text{if } l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0 \\ 0, & \text{if } l_w(f_w) \cdot l_{-w}(f_{-w}) = 0 \end{cases},$$

where $l_w = i_w^*l_v$, form a compatible solution to the BKN-equation (see sect. 3.1). It is straightforward to check that the solution satisfies the condition (A) of Theorem 3.1.

Conversely, let (a, γ) be a compatible solution to the BKN-equation satisfying the condition (A) of Theorem 3.1. For every vertex $v \in V$, choose a Waldhausen basis $\{z_w, f_w\}_{w \in \partial v}$ of M_v . Then for the coefficients of the decomposition $f_{-w} = b_w z_w + d_w f_w$ we have: b_w is the intersection index for the edge $w \in \partial v$, and $k_v = \sum_{w \in \partial v} \frac{d_w}{b_w}$ is the charge of the vertex v (see Lemma 1.1 and sect. 1.3.2). For an edge $w \in \partial v$, define a class $l_w \in H^1(T_{|w|}; \mathbb{R})$ by $l_w(f_w) := a_v$ and $l_w(z_w) := \frac{\gamma_w}{|b_w|}a_{w^+} - \frac{d_w}{b_w}a_v$. Then

$$\sum_{w \in \partial v} l_w(z_w) = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|}a_{w^+} - k_v a_v = 0,$$

thus the collection of the classes $\{l_w : w \in \partial v\}$ satisfies the conditions of Lemma 5.1. By that Lemma, there is a class $l_v \in H^1(M_v; \mathbb{R})$ for which $i_w^*l_v = l_w$ for all $w \in \partial v$. Since $l_w(f_{-w}) = \text{sign}(b_w)\gamma_w a_{w^+}$ and $|\gamma_w| \leq 1$, for the vertex $v' = w^+$ we have $|l_w(f_{-w})| \leq a_{v'} = l_{-w}(f_{-w})$. Assume that the equalities $|l_{-w}(f_w)| = l_w(f_w)$ and $|l_w(f_{-w})| = l_{-w}(f_{-w})$ hold. If $l_w(f_w) = 0 =$

$l_{-w}(f_{-w})$, then $l_w = l_{-w} = 0$. Otherwise $l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0$, because the solution is compatible. Then $|\gamma_w| = |\gamma_{-w}| = 1$, and hence $\gamma_{-w} = \gamma_w$, because, we recall, $\gamma_w \cdot \gamma_{-w} \neq -1$. It easily follows that $l_{-w} = \pm l_w$. Therefore, the cohomological classes $\{l_v : v \in V\}$ are compatible. It is straightforward to check that these classes satisfy the condition (A) of Theorem 2.3. \square

Proposition 5.2 gives an additional flexibility for the proof of Theorems 2.3 and 3.1: it is convenient to use compatible cohomological classes in some cases, and compatible solutions to BKN-equations in other ones.

5.2 Local conditions of extension

Here we establish necessary and sufficient boundary conditions for every maximal block to have an extension to a NPC-metric on it (Lemma 5.3), and to have an extension to a horizontal immersion in it (Lemma 5.6). These conditions are an essential part of the proof of Theorems 2.3 and 3.1. Though Lemmas 5.3 and 5.6 are formulated differently and their proofs are technically different, there is an explicit analogy between them.

Lemma 5.3. *Assume that for a maximal block M_v of a manifold $M \in \mathfrak{M}$ an orientation of its Seifert fibers is fixed and a flat metric on the boundary ∂M_v is given, i.e., for every $w \in \partial v$ a positive definite quadratic form g_w on $H_1(T_{|w|}; \mathbb{R})$ is given, and*

- (1) $g_w(f_w, f_w) = a_v^2 > 0$ is independent of $w \in \partial v$;
- (2) $\sum_{w \in \partial v} g(z_w, f_w) = 0$,

where $\{z_w, f_w\}_{w \in \partial v}$ is a Waldhausen basis of the block M_v . Then there is a NPC-metric g_v on M_v which extends the given one on the boundary ∂M_v , i.e., every boundary torus $T_{|w|}$, $w \in \partial v$, is flat and geodesic w.r.t. g_v , and the metric g_v induces on $T_{|w|}$ the metric g_w .

Remark 5.4. Condition (2) is independent of the choice of a Waldhausen basis; Conditions (1) and (2) are also necessary for the existence of a NPC-metric on M_v with flat geodesic boundary (see Lemmas 2.2 and 5.1).

Sketch of the proof of Lemma 5.3. Given $w \in \partial v$, define $l_w \in H^1(T_{|w|}; \mathbb{R})$ by $l_w(a) = \frac{g_w(a, f_w)}{\sqrt{g_w(f_w, f_w)}}$, $a \in H_1(T_{|w|}; \mathbb{R})$. In other words, the class l_w acts as the projection onto $\mathbb{R} \cdot f_w$ w.r.t. the scalar product g_w (cf. the proof of Lemma 2.2). It follows from (1) and (2) that the collection $\{l_w : w \in \partial v\}$ satisfies the conditions of Lemma 5.1 about extension, and thus there is a class $l_v \in H^1(M_v; \mathbb{R})$, which induces a collection $\{l_w\}$, $i_w^* l_v = l_w$, on the

boundary ∂M_v . The class l_v defines a homomorphism $\varphi_v : \pi_1(M_v) \rightarrow \mathbb{R}$ using which we shall construct a representation $\psi_v : \pi_1(M_v) \rightarrow \text{Iso}(\mathbb{H}^2 \times \mathbb{R})$ of the fundamental group of the block M_v in the isometry group of the space $\mathbb{H}^2 \times \mathbb{R}$.

Now, we assume for simplicity that the block M_v is the trivial S^1 -bundle over a compact surface F_v of negative Euler characteristic. Fix a trivialization $M_v = F_v \times S^1$ and an orientation of the surface F_v , what is equivalent to the choice of a Waldhausen basis $\{z_w, f_w\}_{w \in \partial v}$. In this case, the elements z_w , $w \in \partial v$ represent corresponding oriented components of the boundary ∂F_v .

Recall that, since the Euler characteristic of F_v is negative, given any positive numbers L_w , $w \in \partial v$, there is a metric of constant curvature -1 with geodesic boundary on the surface F_v such that the length of w -component equals L_w .

Now, as L_w , $w \in \partial v$, we take the length of z_w projected on the orthogonal to f_w direction w.r.t. the metric g_w . This defines as above a hyperbolic metric on the surface F_v and hence a representation $\eta_v : \pi_1(F_v) \rightarrow \text{Iso}(A_v)$ of the group $\pi_1(F_v)$ in the isometry group of the universal cover $A_v \subset \mathbb{H}^2$ of the surface F_v .

We define the representation $\psi_v : \pi_1(M_v) \rightarrow \text{Iso}(A_v \times \mathbb{R})$ as $\psi_v(\gamma) = (\eta_v \circ \pi_v(\gamma), \varphi_v(\gamma))$ for $\gamma \in \pi_1(M_v)$, where $\pi_v : \pi_1(M_v) \rightarrow \pi_1(F_v)$ is the projection homomorphism onto the first factor. It is easily to check that its image acts discretely and freely on the metric product $A_v \times \mathbb{R}$, and this gives a required NPC-metric on the block M_v .

In the general case the argument is similar: one needs to use that there is a hyperbolic structure on the orbifold $\mathcal{O}_v = \mathcal{O}M_v$ with prescribed lengths L_w of the boundary components, what gives a representation $\eta_v : \pi_1(\mathcal{O}_v) \rightarrow \text{Iso}(A_v)$, and then to take the homomorphism π_v from the exact sequence

$$1 \longrightarrow \mathbb{Z} \cdot f_v \longrightarrow \pi_1(M_v) \xrightarrow{\pi_v} \pi_1(\mathcal{O}_v) \longrightarrow 1$$

of the Seifert bundle for M_v . □

Remark 5.5. The NPC-metric constructed above on the trivial bundle $M_v \simeq F_v \times S^1$ typically is not a metric product: the holonomy of the circle S^1 along some noncontractible loops in F_v can be nontrivial. A metric is the product if it is possible to find a trivialization $z \in H_1(\partial M_v; \mathbb{Z})$, $z = \bigoplus_{w \in \partial v} z_w$, such that the element z_w is orthogonal to f_w w.r.t. the metric g_w for all $w \in \partial v$.

Lemma 5.6. *Assume that for an oriented block M_v of a manifold $M \in \mathfrak{M}$ an orientation of its Seifert fibers is fixed and for every $w \in \partial v$ two elements c_w^+ , $c_w^- \in H_1(T_{|w|}; \mathbb{Z})$ are given such that $\alpha_w^\pm = c_w^\pm \wedge_w f_w \geq 0$, the class $c_w = c_w^+ + c_w^-$ is even, i.e., $\frac{1}{2}c_w \in H_1(T_{|w|}; \mathbb{Z})$, and the following conditions hold*

(1) $c_w \wedge_w f_w = a_v > 0$ is independent of $w \in \partial v$;

(2) $\sum_{w \in \partial v} z_w \wedge_w c_w = 0$,

where $\{z_w, f_w\}_{w \in \partial v}$ is a Waldhausen basis of the block M_v . Then there is an integer $d(M_v) \geq 1$ such that for any integer multiple $d \geq 1$ of $d(M_v)$ there is a horizontal immersion $g_v : S_v \rightarrow M_v$ of a compact surface S_v with boundary $\partial S_v = \cup_{w \in \partial v} (\gamma_w^+ \cup \gamma_w^-)$, the union of the connected components, such that $[g_v(\gamma_w^\pm)] = d \cdot c_w^\pm$ for all $w \in \partial v$, for which $\alpha_w^+ \cdot \alpha_w^- \neq 0$. In the case $\alpha_w^+ \cdot \alpha_w^- = 0$ for some $w \in \partial v$, one holds $[g_v(\gamma_w^+)] = [g_v(\gamma_w^-)] = \frac{d}{2} \cdot c_w$.

Remark 5.7. Condition (2) is independent of the choice of a Waldhausen basis; Conditions (1) and (2) are also necessary for the existence of an immersion $g_v : S_v \rightarrow M_v$ with prescribed behavior on the boundary (see Lemmas 2.1 and 5.1).

The proof of Lemma 5.3 was based on the existence of a hyperbolic metric with prescribed lengths of the boundary components on a given surface (orbifold) of negative Euler characteristic. The proof of Lemma 5.6 is based on a similar existence property for coverings of a given surface of negative Euler characteristic with prescribed behavior on the boundary. Here is the precise statement.

Lemma 5.8. *Let F be a compact orientable surface of positive genus with nonempty boundary, W be the set of its boundary components. Assume that every boundary component $w \in W$ is covered by a collection S_w consisting of $n_w \geq 1$ circles with multiplicities $a_w^s \geq 1$, $s \in S_w$, where the sum $\sum_{s \in S_w} a_w^s = a$ is independent of w and the number of the circles $\sum_w n_w$ has the same parity as $a \cdot |W|$. Then the covering of the boundary ∂F can be extended to a covering $S \rightarrow F$ of multiplicity a by a connected surface S .*

Proof. It suffices to find a homomorphism of the group $\pi_1(F)$ into the permutation group of a fiber (having cardinality a) with transitive image, which maps the boundary components into permutations with the prescribed cycle structure. Due to the parity condition, the product of the boundary cycles is an even permutation, and thus by a result from [JLLP], it is the commutator of an n -cycle and an involution. Now, since the genus of F is positive, the required homomorphism does exist. \square

Proof of Lemma 5.6. First, we assume for simplicity that M_v is the trivial S^1 -bundle and that the trivialization $M_v = F_v \times S^1$ corresponds to the chosen Waldhausen basis. Taking if necessary a covering of F_v with multiplicity $d(M_v) := |\partial v|$ and with the same number of the boundary components, we can assume that the genus of F_v is positive.

In the decompositions $c_w^\pm = \alpha_w^\pm z_w + \beta_w^\pm f_w$, the integer $\alpha_w^\pm = c_w^\pm \wedge_w f_w$ are nonnegative, $\alpha_w^+ + \alpha_w^- = a_v > 0$ is even for all $w \in \partial v$, and $\sum_{w \in \partial v} (\beta_w^+ + \beta_w^-) = 0$. We construct a required immersion $g_v : S_v \rightarrow F_v \times S^1$ as $g_v = (g_v^{\text{hor}}, g_v^{\text{vert}})$. Here $g_v^{\text{hor}} : S_v \rightarrow F_v$ is a covering of the degree a_v , which on a boundary component $\gamma_w^\pm \subset \partial S_v$ covers the corresponding component of ∂F_v with degree α_w^\pm , if both α_w^+ , α_w^- are nonzero, and both components γ_w^\pm cover the corresponding component of ∂F_v with degree $a_v/2$, if $\alpha_w^+ \cdot \alpha_w^- = 0$. Lemma 5.8 provides such a covering with a connected surface S_v . The parity condition is fulfilled since the number of the boundary components of S_v is twice of those of F_v , and a_v is even.

We require that the map $g_v^{\text{vert}} : S_v \rightarrow S^1$ has the degree β_w^\pm on the boundary component γ_w^\pm , if $\alpha_w^+ \cdot \alpha_w^- \neq 0$, and the degree $\frac{1}{2}(\beta_w^+ + \beta_w^-)$ on both boundary components γ_w^+ , γ_w^- , if $\alpha_w^+ \cdot \alpha_w^- = 0$ ($\beta_w^+ + \beta_w^-$ is even because the class c_w is even). Due to the connectedness of S_v and the condition $\sum_{w \in \partial v} (\beta_w^+ + \beta_w^-) = 0$, such a map does exist since $[S_v, S^1] = H^1(S_v; \mathbb{Z})$. Now, the immersion g_v has the required properties by the construction.

In the general case, we remove from M_v the interiors of fibered solid tori (with disjoint closures), which are boundaries of the singular fibers, and denote by $\partial^s v$ the set of the new boundary components of the complement M'_v to the removed pieces. The manifold M'_v is a trivial S^1 -bundle over a compact orientable surface F'_v of negative Euler characteristic, furthermore, for every torus $T_w \subset \partial M'_v$, $w \in \partial^s v$, the class $c_w \in H_1(T_w; \mathbb{Z})$ of its meridian, i.e., a simple, closed curve bounding a disc in the corresponding removed solid tori, is fixed. We assume that the orientations of the meridians are chosen such that $\alpha_w := c_w \wedge_w f_w > 0$ for all $w \in \partial^s v$. We put $a'_v = a_v \cdot \prod_{w \in \partial^s v} \alpha_w$, $\partial^* v = \partial v \cup \partial^s v$, and for $w \in \partial^* v$ let $d_w = a'_v / \alpha_w \in \mathbb{Z}$, where $\alpha_w = a_v$ for $w \in \partial v$. Then all integer classes $c'_w = d_w c_w$, $w \in \partial^* v$ have one and the same intersection index with regular fiber f_w , $c'_w \wedge_w f_w = a'_v$. Fix a trivialization $M'_v = F'_v \times S^1$. Then for the corresponding Waldhausen basis $\{z_w, f_w\}_{w \in \partial^* v}$ we have $\sum_{w \in \partial^* v} z_w \wedge_w c'_w = 0$.

For the collection of the classes $c_w'^\pm = d \cdot c_w^\pm$, $w \in \partial v$, where $d = d(M_v) := a'_v / a_v$ (is multiplied by $|\partial^* v|$ if necessarily for positivity of the genus), and c'_w , $w \in \partial^s v$, we construct an immersion $g'_v : S'_v \rightarrow M'_v$, $g'_v = (g_v^{\text{hor}}, g_v^{\text{vert}})$ as above with only distinction that every boundary component $w \in \partial^s v$ of the surface F'_v is covered by d_w boundary components of the surface S'_v with degree α_w on each one under the covering g_v^{hor} . The parity condition is reduced to that the number $\sum_{w \in \partial^s v} d_w$ has to be even, and it is fulfilled since this sum is an integer multiple of even a_v . For the map $g_v^{\text{vert}} : S'_v \rightarrow S^1$, an additional condition is that it has the degree $z_w \wedge_w c'_w$ on each of d_w corresponding boundary components of S'_v for every $w \in \partial^s v$.

One can assume that the image $g'_v(S'_v) \subset M'_v$ meets every boundary com-

ponent $w \in \partial^s v$ over a collection of d_w curves parallel to the meridian. Attaching the discs to these curves in the corresponding solid tori, we obtain the required immersion g_v for $d = d(M_v)$. Now, the case of an arbitrary multiple d of $d(M_v)$ easily follows. \square

5.2.1 Historical remarks

The statement close to Lemma 5.3 is given in [L]. Lemma 5.6 is a particular case of Lemma 3.1 from [N3]. Lemma 5.8 and its proof are taken from [N3].

5.3 Properties (I), (HI), (E), (F), (NPC)

5.3.1 Existence of compatible cohomological classes

Here we show that any of the properties (I), (HI), (E), (F), (NPC) for a manifold $M \in \mathfrak{M}$ implies the existence of a compatible cohomological classes satisfying the corresponding condition of Theorem 2.3. To this end, we use constructions of sect. 2.1. For the virtual properties (VE) and (VF) this step of the proof is more difficult, and it is discussed in sect. 5.4.

Proposition 5.9. *Any of the properties (I), (HI), (E), (F), (NPC) of a manifold $M \in \mathfrak{M}$ implies the existence of a compatible collection of cohomological classes satisfying the corresponding condition of Theorem 2.3.*

Proof. For (I) and (NPC) this follows from Lemmas 2.1 and 2.2 respectively.

(HI) If an immersion $g : S \rightarrow M$ is horizontal, then for the corresponding compatible cohomological classes $\{l_v : v \in V\}$ we have $l_v(f_v) > 0$ by the construction.

(E) Let $g : S \rightarrow M$ be a π_1 -injective embedding. Consider the corresponding compatible collection $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ and assume that $l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0$ for some edge $w \in W$, where $l_w = i_w^* l_v$. Then the embedding g is horizontal in both blocks $M_v, M_{v'} \subset M$, $v = w^-$, $v' = w^+$. In this case, the intersection $g(S) \cap T_{|w|}$ consists of a collection of parallel circles. Hence, they all are consistent or not simultaneously. Thus $l_w = \pm l_{-w}$.

(F) Let $p : M \rightarrow S^1$ be a fibration. We fix a fiber $S = p^{-1}(t)$, $t \in S^1$. One can assume that the surface $S \subset M$ is horizontal. As usual, using this surface, we define compatible cohomological classes $\{l_v : v \in V\}$. It follows from horizontality that $l_v(f_v) > 0$ for all $v \in V$. We fix a generator α of the group $H^1(S^1; \mathbb{Z})$ and put $\varepsilon_v = \text{sign } \alpha(p_*(f_v)) \in \{\pm 1\}$. Then $l_v = \varepsilon_v p_v^* \alpha$, where p_v is the restriction of p to the maximal block M_v . Thus for every edge $w \in W$ we have

$$l_{-w}(f_w) = i_{-w}^* l_{v'}(f_w) = i_{-w}^* \varepsilon_{v'} p_{v'}^* \alpha(f_v) = \varepsilon_{v'} p_v^* \alpha(f_v) = \varepsilon_{v'} \varepsilon_v l_w(f_w),$$

where v, v' are the vertices of w . \square

5.3.2 Finding a NPC-metric on a graph-manifold

To complete the proof of Theorems 2.3 and 3.1 for the property (NPC), it remains to construct a NPC-metric on M , if a compatible collection of cohomological classes $\{l_v : v \in V\}$ satisfying the condition (NPC) of Theorem 2.3 is given. We do that using Lemma 5.3.

For $w \in \partial v$ we put $l_w = i_w^* l_v \in H^1(T_{|w|}; \mathbb{R})$ and define a symmetric bilinear form g_w on $H_1(T_{|w|}; \mathbb{R})$ by $g_w(f_w, f_w) = (l_w(f_w))^2$, $g_w(f_{-w}, f_{-w}) = (l_{-w}(f_{-w}))^2$, and $g_w(f_w, f_{-w}) = l_w(f_w) \cdot l_{-w}(f_{-w})$. It follows from the (NPC)-condition of compatibility $|l_{-w}(f_w)| < l_w(f_w)$ that the form g_w is positive definite. Furthermore, it obviously satisfies the condition (1) of Lemma 5.3. The condition (2) of the same Lemma is also fulfilled since the forms g_w are defined via the classes l_v . By Lemma 5.3, we obtain a NPC-metric on every maximal block M_v . It follows from the (NPC)-condition of symmetry $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ that the constructed metrics coincide on the gluing tori (cf. the proof of Lemma 2.2). Therefore, we obtain a NPC-metric on M . (This metric is C^1 -smooth on M and real analytic inside of every block M_v being modelled on $H^2 \times \mathbb{R}$. Its curvature is nonpositive in Alexandrov's sense. The metric can easily be smoothed to an C^∞ -smooth NPC-metric (see, for example, [L]), though. \square

5.3.3 Rational approximation

To implement the properties (I) – (VF), one needs to have compatible collections of integer cohomological classes while we have only real ones. Thus we first approximate real classes by rational ones, and then we obtain integer classes from them. It is convenient to construct an approximation in terms of solutions to the BKN-equation. The next proposition is sufficient for the properties (I), (HI), (E), (F). To implement the virtual properties (VE), (VF) one needs to have a symmetric approximating angle function γ' , and a symmetric approximation is achieved by another method, see sect. 5.8.

Proposition 5.10. *Let (a, γ) be a (real) compatible solution to the BKN-equation over $M \in \mathfrak{M}$. Then it can be approximated by rational compatible solutions (a', γ') either so that*

- $\gamma' = \gamma$ and $a'_v = a_v$ for the vertices $v \in V$ for which $a_v = 0$, if the angle function γ takes values in $\{0, \pm 1\}$;

or so that

- $a'_v > 0$ for all $v \in V$ and $|\gamma'_w| < 1$ for all edges $w \in W$, if $|\gamma_{w_0}| < 1$ for some edge $w_0 \in W$.

In other words, if the angle function γ takes values on $\{0, \pm 1\}$ and $\gamma_w = 0$ for some edge $w \in W$, then we have two types of a rational approximation of the solution (a, γ) . The approximation described in the first part will be used for the properties (I), (HI), (E) and (F). The approximation described in the second part will be used for the properties (I) and (HI).

The first part is easy to prove: unknown lengths a_v , $v \in V$, when (rational) angles γ are fixed, satisfy a linear homogeneous system of $|V|$ equations with rational coefficients and the zero determinant, which follows from the BKN-equation. By a classical theorem from linear algebra, the solutions to such a system can be parameterized by linear functions with rational coefficients. This easily implies a required approximation. The proof of the second part we start with construction of perturbations of the initial solution (in the class of solutions) do not caring first about the rationality. This is achieved in the following three lemmas.

Lemma 5.11. *Let (a, γ) be a compatible solution to the BKN-equation. Then it can be approximated by compatible solutions (a', γ') so that $a'_v > 0$ for all $v \in V$.*

Proof. If the length function a has zeroes on V , then since the solution is nontrivial there is a vertex $v \in V$ with $a_v > 0$ and $a_{v'} = 0$ for a neighboring vertex v' . Consider an edge $w \in \partial v$ coming in v' . Recall that then $\gamma_{w'} = \gamma_{-w'} = 0$ for all $w' \in \partial v'$ since the solution (a, γ) is compatible. We choose an arbitrarily small $a'_{v'} > 0$ and find γ'_{-w} with $k_{v'} a'_{v'} = \frac{\gamma'_{-w}}{|b_{-w}|} a_v$. Since the angle function γ vanishes on all edges going out of the vertex v' , we achieve by this the BKN-equation over v' for the perturbed functions a', γ' . The BKN-equation for them is also fulfilled over the vertex v because the function γ vanishes on all edges pointing out from v to v' . The procedure obviously leads to a required approximation. \square

Using this Lemma, we assume from now on that the length function a is positive.

Lemma 5.12. *Let (a, γ) be a compatible solution to the BKN-equation. Assume that for a vertex $v \in V$ there is an edge $w_0 \in \partial v$ with $|\gamma_{w_0}| < 1$. Then (a, γ) can be approximated by compatible solutions (a', γ') so that $|\gamma'_w| < 1$ for all $w \in \partial v$.*

Proof. Unknown angles γ_w , $w \in \partial v$ enter only the BKN-equation over the vertex v :

$$k_v a_v = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w+}.$$

Thus changing only them and keeping this equality hold, we obtain a new solution to the BKN-equation. One needs to change only those γ_w , $w \in \partial v$, for which $|\gamma_w| = 1$. Using that the length function a is positive, their absolute values can be (arbitrarily small) decreased at the expense of γ_{w_0} . \square

Lemma 5.13. *Let (a, γ) be a compatible solution to the BKN-equation. Assume that $|\gamma_w| < 1$ for all $w \in \partial v$ for some vertex $v \in V$. Then (a, γ) can be approximated by compatible solutions (a', γ') so that $|\gamma'_{\pm w}| < 1$ for all $w \in \partial v$.*

Proof. We change the angle function γ on every edge $-w$ with $|\gamma_{-w}| = 1$ and $w \in \partial v$, and the length function a on the vertex v so that $|\gamma'_{-w}| \cdot a'_v = |\gamma_{-w}| \cdot a_v$, where the ratio $\frac{a'_v}{a_v} > 1$ is arbitrarily close to 1 (here $'$ means taking approximating functions). By that one achieves the property $|\gamma'_{-w}| < 1$ for all edges $w \in \partial v$ with the BKN-equation satisfied for all final vertices of these edges (except the loops). To fulfill the BKN-equation for the initial vertex v , we distribute the excess $k_v(a'_v - a_v)$ over the edges $w \in \partial v$, changing in an appropriate way the values γ_w . One can do this keeping the condition $|\gamma_w| < 1$, because the value $|k_v(a'_v - a_v)|$ can be chosen arbitrarily small. \square

Proof of Proposition 5.10. As it was explained above, to prove the first part of the Proposition, one can use a rational parameterization of the solutions to the BKN-equation obtained when rational values of the angle function γ are fixed. It follows from compatibility of the solution (a, γ) that the equations over the vertices $v \in V$ with $a_v = 0$ put no restriction on any approximating function a' having the same zeroes as the function a . Thus it remains to prove the second part of the Proposition.

It suffices to approximate the solution (a, γ) by rational solutions (a', γ') , satisfying the conditions of the Proposition. Then the solutions (a', γ') sufficiently close to (a, γ) will be compatible automatically. According Lemmas 5.11, 5.12, 5.13 one can assume that the initial solution (a, γ) to the BKN-equation satisfies the conditions $a_v > 0$ for all vertices $v \in V$ and $|\gamma_w| < 1$ for all edges $w \in W$.

Now, we approximate the solution (a, γ) by solutions (a', γ') , where the length function a' takes rational values. Fix $v \in V$ and approximate the number a_v by rational numbers $a'_v > 0$. The excess $k_v(a'_v - a_v)$ from the BKN-equation over the vertex v we distribute between the summands $\frac{\gamma_w}{|b_w|} a_{w+}$, $w \in \partial v$, changing only if necessarily appropriate γ_w , keeping the condition $|\gamma_w| < 1$ and leaving invariant the numbers a_{w+} . By this, one achieves the BKN-equation to hold over the vertex v . For any v' , neighboring to v , only summands from the right hand side of the BKN-equation over v' , corresponding to the edges $w \in \partial v'$ with $-w \in \partial v$ are changed during the described manipulations. We compensate these changes by the condition

$\gamma'_w \cdot a'_v = \gamma_w \cdot a_v$ for every such edge w . We have $|\gamma'_w| < 1$, if the ratio $\frac{a_v}{a'_v}$ is sufficiently close to 1. Therefore, one can assume that the length function a from the initial compatible solution (a, γ) takes positive rational values on all vertices $v \in V$. Now, it is easy to approximate the numbers γ_w by rational numbers γ'_w for any edge $w \in W$ fulfilling the BKN-equation. \square

5.3.4 Finding a horizontal immersion

To complete the proof of Theorems 2.3 and 3.1 for the properties (I) and (HI), we construct an immersed horizontal surface in M , if there is a compatible collection of cohomological classes on M . Using Proposition 5.10, we find a compatible rational solution (a, γ) to the BKN-equation for which $a_v > 0$ for all $v \in V$, and if $|\gamma_{w_0}| = 1$ for some edge $w_0 \in W$, then $\gamma_{-w} = \gamma_w = \pm 1$ for all edges $w \in W$. This gives an integer collection $\{l_v \in H^1(M_v; \mathbb{Z}) : v \in V\}$ of compatible classes, which satisfies the following, more strong, condition of compatibility: if $|l_{-w_0}(f_{w_0})| = l_{w_0}(f_{w_0})$ for some edge $w_0 \in W$, then $l_{-w} = \pm l_w$ for all edges $w \in W$.

For every edge $w \in W$ we put $l_w^+ = l_w + l_{-w}$, $l_w^- = l_w - l_{-w}$ and consider the elements $c_w^+, c_w^- \in H_1(T_{|w|}; \mathbb{Z})$ dual to l_w^+, l_w^- . These elements satisfy the conditions of Lemma 5.6 because:

- $\alpha_w^\pm = c_w^\pm \wedge f_w = l_w(f_w) \pm l_{-w}(f_w) \geq 0$;
- the class $c_w = c_w^+ + c_w^-$ is even being dual to the class $2l_w$;
- $c_w \wedge f_w = 2l_w(f_w) > 0$ for all $v \in V$ and $w \in \partial v$; the condition (2) of Lemma 5.6 is also holds since the classes l_w , $w \in \partial v$ are boundary values for the class l_v , $i_w^* l_v = l_w$, see Lemma 5.1.

Moreover, the condition of the strong compatibility means that either $\alpha_w^+ \cdot \alpha_w^- \neq 0$ for all $w \in W$ or $\alpha_w^+ \cdot \alpha_w^- = 0$ for all $w \in W$. Since $c_{-w}^+ = c_w^+$, $c_{-w}^- = -c_w^-$ and if $\alpha_w^+ \cdot \alpha_w^- = 0$, then $c_{-w} = \pm c_w$, horizontal immersions $g_v : S_v \rightarrow M_v$ provided by Lemma 5.6 for each maximal block M_v can be easily pasted on every gluing torus $T_{|w|}$ at the expense of an appropriate multiplicity d universal for all blocks. This gives a required horizontal immersion $g : S \rightarrow M$. \square

Corollary 5.14. *The property (I) implies the property (HI).* \square

5.3.5 Finding a fibering over the circle

To complete the proof of Theorems 2.3 and 3.1 for the property (F), we construct a fibration $M \rightarrow S^1$ with fiber which is a closed surface of negative Euler characteristic, if there are compatible cohomological classes $\{l_v : v \in V\}$ on M satisfying the condition (F) of Theorem 2.3. This condition implies that $\varepsilon_v l_w = \varepsilon_{v'} l_{-w}$ for every edge $w \in W$ from v to v' . According to the

first part of Proposition 5.10, one can assume the collection $\{l_v : v \in V\}$ to be integer. Using the canonical isomorphism $H^1(M_v; \mathbb{Z}) = [M_v, S^1]$ and that $l_v(f_v) \neq 0$, we find for every maximal block M_v a map $M_v \rightarrow S^1$ corresponding to the class $\varepsilon_v l_v$ of nonzero degree on the Seifert fibers for M_v . It is well known that such a map is homotopic to a fibration $p_v : M_v \rightarrow S^1$ (see the discussion of this question in [N2]). Since $\varepsilon_v l_w = \varepsilon_{v'} l_{-w}$ for any edge $w \in W$ from v to v' , we can choose the fibrations p_v and $p_{v'}$ coinciding on the gluing torus $T_{|w|}$. This gives a required fibration $p : M \rightarrow S^1$.

5.3.6 Finding an embedded surface

We consider now the implementation of the property (E). By the first part of Proposition 5.10, one can assume that all classes $l_v \in H^1(M_v; \mathbb{Z})$, $v \in V$ of the corresponding compatible collection are even. Given a block M_v with $l_v(f_v) \neq 0$, we find as above a fibration $M_v \rightarrow S^1$. Its fibers are horizontally embedded surfaces whose relative class corresponds to the class l_v under the isomorphism $H_2(M_v, \partial M_v; \mathbb{Z}) \simeq H^1(M_v; \mathbb{Z})$. If $l_v(f_v) \cdot l_{v'}(f_{v'}) \neq 0$ for neighboring vertices v, v' , then $l_w = \pm l_{-w}$ by the condition (E) of Theorem 2.3, where the edge $w \in W$ points out of v to v' . Thus one can glue corresponding horizontal surfaces in the blocks $M_v, M_{v'}$ on the separating torus $T_{|w|}$ (slightly deforming them if necessarily). This gives a surface S' horizontally embedded into the union of all maximal blocks M_v with $l_v(f_v) \neq 0$.

Assume now that $l_v(f_v) = 0$ for some vertex $v \in V$. We can further assume that $l_{v'}(f_{v'}) \neq 0$ for a neighboring vertex v' . Consider an edge $w \in W$ from v to v' . The compatibility condition $|l_{-w}(f_w)| \leq l_w(f_w) = 0$ implies that the horizontal surface $S' \cap M_{v'}$ meets the torus $T_{|w|}$ by a collection of circles parallel to the Seifert fibers of $M_{v'}$. It follows from the parity condition that the number of the circles is even. It is not difficult to close these circles by vertically embedded annuli in M_v so that the annuli are not parallel to the block boundary and pairwise disjoint (taking into account all edges $w \in \partial v$). We leave details to the reader. The described procedure obviously gives a closed surface embedded into M , whose intersection with any block either is horizontal or consists of vertical annuli, or is empty. It is known (see [RW, Lemma 3.4]) that any such an embedding is π_1 -injective. \square

This completes the proof of Theorems 2.3 and 3.1 for all properties except the virtual ones, (VE) and (VF).

5.4 Virtual properties: necessary conditions

We start the proof of Theorems 2.3, 3.1 for the virtual properties (VE) and (VF). The proof of necessity of the corresponding conditions is based on the existence of characteristic coverings of the graph-manifolds and Descent-Lemma 5.16. This Lemma also implies the existence of a NPC-metric on a graph-manifold if some its finite cover carries a NPC-metric. This fact has been proved earlier by another method in [KL] for a wider class of manifolds.

5.4.1 Descent Lemma

Let $s \geq 1$ be an integer. A covering $p : \tilde{T} \rightarrow T$ of tori is called s -characteristic, if the image of the induced homomorphism is $p_*(H_1(\tilde{T}; \mathbb{Z})) = s \cdot H_1(T; \mathbb{Z})$. A covering $\tilde{M} \rightarrow M$ of manifolds of the class \mathfrak{M} is called s -characteristic, if its restriction to each JSJ-torus in \tilde{M} is s -characteristic.

The existence of s -characteristic coverings of a manifold $M \in \mathfrak{M}$ with an arbitrary multiple s of some s_0 has been proved in [LW] (by arguments in spirit of the proof of Lemma 5.6), see also [N2]. Furthermore, the covering can be pushed through any given finite covering $\widehat{M} \rightarrow M$.

Characteristic covering are good by that they allow easily to trace the behavior of the intersection indices and the charges. Namely, let $\tilde{M} \rightarrow M$ be a s -characteristic covering of manifolds of the class \mathfrak{M} . Denote by $p : \tilde{\Gamma} \rightarrow \Gamma$ the induced map of the graphs $\tilde{\Gamma} = \Gamma_{\tilde{M}}$, $\Gamma = \Gamma_M$. The restriction $\tilde{M}_{\tilde{v}} \rightarrow M_v$ of the covering to any maximal block $\tilde{M}_{\tilde{v}} \subset \tilde{M}$ has the degree $d_{\tilde{v}} \cdot s^2$, the number $s^2 \cdot \sum_{p(\tilde{v})=v} d_{\tilde{v}}$ is independent of the vertex $v \in V$ of the graph Γ and it equals the degree of the covering $\tilde{M} \rightarrow M$.

We assume that an orientation of M and fiber orientations of its maximal blocks are fixed, and for \tilde{M} the orientations induced by the covering are chosen (it can be done because M contains no one-sided Klein bottles). It follows from the characteristic property that the intersection indices for the edges $\tilde{w} \in \tilde{W}$ of $\tilde{\Gamma}$ and $w = p(\tilde{w}) \in W$ of Γ coincide, $b_{\tilde{w}} = b_w$. For the charges of $\tilde{v} \in \tilde{V}$ and $v = p(\tilde{v}) \in V$ one holds

$$k_{\tilde{v}} = d_{\tilde{v}} \cdot k_v,$$

what easy to see from the fact that every boundary torus T_w of the block M_v is s -characteristically covered by $d_{\tilde{v}}$ tori of the block $\tilde{M}_{\tilde{v}}$, and thus their contribution into the charge $k_{\tilde{v}}$ is $d_{\tilde{v}}$ -fold of the contribution of T_w into the charge k_v (one can find details in [LW], [N2]). These two properties immediately imply the following Lemma.

Lemma 5.15. *Let $p : \widetilde{\Gamma} \rightarrow \Gamma$ be the map of the graphs $\widetilde{\Gamma} = \Gamma_{\widetilde{M}}$, $\Gamma = \Gamma_M$, of manifolds M , $\widetilde{M} \in \mathfrak{M}$ induced by a characteristic covering $\widetilde{M} \rightarrow M$. Then for any solution (a, γ) to the BKN-equation over M the functions $\widetilde{a} = a \circ p$, $\widetilde{\gamma} = \gamma \circ p$ form a solution to the BKN-equation over \widetilde{M} . \square*

Furthermore, only these two properties of characteristic coverings will be used in the proof of the following Descent Lemma.

Lemma 5.16. *Let $\widetilde{M} \rightarrow M$ be a characteristic covering of a manifold $M \in \mathfrak{M}$. If the BKN-equation over \widetilde{M} has a compatible solution satisfying one of the conditions (VE), (VF), (NPC) of Theorem 3.1, then the BKN-equation over M has a compatible solution satisfying the same condition.*

Proof. We use notations introduced above for a characteristic covering, and for every vertex $v \in V$ (of the graph of M) we define a quadratic form D_v on $\mathbb{R}^{\widetilde{V}}$,

$$(D_v x, x) = \sum_{p(\widetilde{v})=v} d_{\widetilde{v}} \cdot x_{\widetilde{v}}^2, \quad x \in \mathbb{R}^{\widetilde{V}}.$$

Let $(\widetilde{a}, \widetilde{\gamma})$ be a compatible solution to the BKN-equation over the cover \widetilde{M} . For every edge $w \in W$ (of the graph of M) we define a quadratic form G_w on $\mathbb{R}^{\widetilde{V}}$, putting

$$(G_w x, x) = \sum_{p(\widetilde{w})=w} \widetilde{\gamma}_{\widetilde{w}} x_{\widetilde{w}^-} x_{\widetilde{w}^+}, \quad x \in \mathbb{R}^{\widetilde{V}}.$$

From the condition $|\widetilde{\gamma}_{\widetilde{w}}| \leq 1$, we obtain $(G_w x, x)^2 \leq (D_v x, x)(D_{v'} x, x)$ for every $x \in \mathbb{R}^{\widetilde{V}}$, where $v = w^-$ is the initial vertex of the edge w , $v' = w^+$ its final vertex. To deduce the last inequality, one needs to apply twice the Cauchy-Schwartz inequality and to use that the number of edges $\widetilde{w} \in \partial \widetilde{v}$ projected onto an edge w , $p(\widetilde{w}) = w$, is equal to $d_{\widetilde{v}}$ for every vertex $\widetilde{v} \in \widetilde{V}$ with $p(\widetilde{v}) = v$.

Using the relations for the intersection indices and the charges between M and its cover \widetilde{M} , and the BKN-equation

$$k_{\widetilde{v}} \widetilde{a}_{\widetilde{v}} = \sum_{\widetilde{w} \in \partial \widetilde{v}} \frac{\widetilde{\gamma}_{\widetilde{w}}}{|b_{\widetilde{w}}|} \widetilde{a}_{\widetilde{w}^+}, \quad \widetilde{v} \in \widetilde{V},$$

for $(\widetilde{a}, \widetilde{\gamma})$, we obtain

$$k_v (D_v \widetilde{a}, \widetilde{a}) = \sum_{w \in \partial v} \frac{1}{|b_w|} (G_w \widetilde{a}, \widetilde{a}), \quad v \in V.$$

Thus the functions $a : V \rightarrow \mathbb{R}$, $\gamma : W \rightarrow \mathbb{R}$, defined by $a_v = (D_v \tilde{a}, \tilde{a})^{1/2}$ (the form D_v is nonnegative),

$$\gamma_w = \frac{(G_w \tilde{a}, \tilde{a})}{a_{w^-} \cdot a_{w^+}}, \quad \text{if } a_{w^-} \cdot a_{w^+} \neq 0,$$

and $\gamma_w = 0$, if $a_{w^-} \cdot a_{w^+} = 0$, make up a compatible solution to the BKN-equation for M : $|\gamma_w| \leq 1$ for all $w \in W$.

To complete the proof, we note that the length function a is positive, if \tilde{a} is, and the angle function γ is symmetric, if $\tilde{\gamma}$ is ($\tilde{\gamma}_{-\tilde{w}} = \tilde{\gamma}_{\tilde{w}}$ for all $\tilde{w} \in \tilde{W}$). Furthermore, if $|\tilde{\gamma}| < 1$, then $|\gamma| < 1$. Hence, each of the properties (VE), (VF), (NPC) from Theorem 3.1 of the BKN-equation over \tilde{M} implies the corresponding property of the BKN-equation over M . \square

5.4.2 Existence of a solution to BKN-equation

Proposition 5.17. *Each of the properties (VE), (VF) of a manifold $M \in \mathfrak{M}$ implies the existence of a compatible solution to the BKN-equation over M , which satisfies the corresponding condition of Theorem 3.1.*

Proof. Assume that a manifold $M \in \mathfrak{M}$ has one of the properties (VE), (VF). There is a finite covering $\tilde{M} \rightarrow M$, such that the manifold \tilde{M} has the corresponding property (E) or (F). One can suppose that the covering is characteristic. Then by Proposition 5.9 and due to the fact that Theorems 2.3, 3.1 are equivalent, the BKN-equation over \tilde{M} has a compatible solution $(\tilde{a}, \tilde{\gamma})$, which satisfies the corresponding property (E) or (F) from Theorem 3.1. In any case, the angle function $\tilde{\gamma}$ is symmetric, and in the (F) case the length function \tilde{a} is positive. Now, Lemma 5.16 provides the existence of a required solution to the BKN-equation over M . \square

5.5 Implementing the virtual properties

Here we prove that the conditions of Theorem 3.1 are sufficient to implement the virtual properties. We start with the key Ascent Lemma.

5.5.1 Ascent Lemma

Let $g : S \rightarrow M$ be a horizontal immersion of a closed surface S into a manifold M of the class \mathfrak{M} , \mathcal{T} be the JSJ-surface in M . Its preimage $\mathcal{T}_g = g^{-1}(\mathcal{T})$ consists of a finite, disjoint collection of simple, closed, noncontractible curves on S . One can assume that every curve $T \subset \mathcal{T}_g$ has as the image a simple, closed curve $g(T)$ on the corresponding torus from \mathcal{T} .

Let Γ_g be the graph dual to the decomposition of the surface S , $S = \cup_{v \in V_g} S_v$, defined by \mathcal{T}_g . In other words, the vertex set V_g of the graph Γ_g is the set of the connected components of the splitting $S|\mathcal{T}_g$ (which are called the blocks S_v of S), and the set of (nonoriented) edges can be identified with the collection \mathcal{T}_g . The set of oriented edges of the graph Γ_g is denoted by W_g , and for the edges $w \in W_g$ we shall use the same agreements and notations as for the oriented edges $w \in W$ of the graph $\Gamma = \Gamma_M$. The immersion g induces the map $\bar{g} : \Gamma_g \rightarrow \Gamma$ of the graphs taking the vertices into the vertices and edges into the edges.

Fix an orientation of M , fiber orientations of its maximal blocks, orientations of the curves $T \subset \mathcal{T}_g$ and assign to each edge $w \in W_g$ the number $d_w = |[g(T)] \wedge_{\bar{g}(w)} f_{\bar{g}(w)}|$, where the curve $T \subset \mathcal{T}_g$ corresponds to the nonoriented edge $(w, -w)$ of the graph Γ_g . In other words, d_w is equal to the degree with which the curve T is mapped onto the corresponding boundary component of the base orbifold $\mathcal{O}M_{v'}$ of the block $M_{v'} \subset M$, into which the block S_{w-} of the surface S , corresponding to the initial vertex of w , is mapped under the composition $\pi_{v'} \circ g$, where $\pi_{v'} : M_{v'} \rightarrow \mathcal{O}M_{v'}$ is the canonical projection.

All oriented cycles of the graph Γ_g we consider below consist of oriented edges of that graph (with compatible orientations).

Lemma 5.18. *The immersion g is a virtual embedding, i.e., $g = \tilde{g} \circ p$ for some finite covering $p : \tilde{M} \rightarrow M$ and a horizontal embedding $\tilde{g} : S \rightarrow \tilde{M}$, if and only if the following condition of cyclic balance holds:*

$$\prod_{w \in c} \frac{d_w}{d_{-w}} = 1$$

for every oriented cycle c in the graph Γ_g .

Sketch of the proof. It is easy to see, using that the maximal blocks of M are Seifert bundles, that the cover space of the covering associated with subgroup $g_*(\pi_1(S)) \subset \pi_1(M)$ is homeomorphic to the product $S \times \mathbb{R}$. One can further assume that the covering $q : S \times \mathbb{R} \rightarrow M$ itself coincides over the zero section $S \times \{0\} \subset S \times \mathbb{R}$ with the immersion g . In what follows, we identify S with $S \times \{0\}$.

For every curve $T \subset \mathcal{T}_g$, the infinite cylinder $T \times \mathbb{R} \subset S \times \mathbb{R}$ covers the corresponding torus $T_{|\bar{g}(w)|} \subset \mathcal{T}$, where $T = (w, -w)$. Since $g(T) \subset T_{|\bar{g}(w)|}$ is a simple closed curve, the preimage $q^{-1}(g(T)) \cap T \times \mathbb{R}$ consists of a countable collection of parallel copies of the curve T . These copies can be labeled by integers $n \in A \subset \mathbb{Z}$ so that any finite subcylinder in $T \times \mathbb{R}$, bounded by T and its copy $T_{w,n}$, is mapped into the torus $T_{|\bar{g}(w)|}$ with degree $|n| \cdot d_w$. We shall say that $T_{w,n}$ is the copy of the level n . Then $T = T_{w,0}$.

The curve T is a boundary component of a block $S_v \subset S$, where $w \in \partial v \subset V_g$. Then its parallel copy $T_{w,n}$ is a boundary component of a copy $S'_v \subset S_v \times \mathbb{R}$ of the block S_v , and $q(S'_v) = g(S_v)$. Using the Seifert bundle structure over the block $M_{\tilde{g}(v)} \subset M$ and that the degree is invariant under a homotopy, it is easy to see that the level of every component of the boundary $\partial S'_v$ is one and the same. Thus we have a collection $S_{v,n}$, $n \in A$, of copies of the block S_v , which lie in the cylinder $S_v \times \mathbb{R}$ with images $q(S_{v,n}) = g(S_v)$ for every vertex $v \in V_g$. Furthermore, for each adjacent vertices $v = w^-$, $v' = w^+$ and copies $S_{v,n}$, $S_{v',n'}$ of the corresponding blocks S_v , $S_{v'}$, adjacent along a common boundary component, we have a key relation

$$n \cdot d_w = n' \cdot d_{-w}, \quad (\text{deg})$$

in which the left and the right hand sides are different representations of the degree of the map of the corresponding cylinder in the torus $T_{|\tilde{g}(w)|}$.

If g is a virtual embedding, then the manifold \widetilde{M} from the corresponding covering $\widetilde{M} \rightarrow M$, which fibers over the circle, can be covered in turn by a product $S \times S^1$, because the holonomy of the gluing of the manifold \widetilde{M} from the cylinder $S \times [0, 1]$ has a finite order (up to an isotopy). Thus the covering $q : S \times \mathbb{R} \rightarrow M$ can be pushed through the covering $S \times S^1 \rightarrow \mathbb{R}$, and hence the preimage $q^{-1}(g(S))$ contains a parallel copy S' of the surface S . Its block decomposition is $S' = \cup_{v \in V_g} S_{v,n(v)}$, and according what was said above, for every edge $w \in W_g$ we have $n_{w^-} \cdot d_w = n_{w^+} \cdot d_{-w}$. Thus for every oriented cycle $c \subset \Gamma_g$ we have

$$\prod_{w \in c} \frac{d_w}{d_{-w}} = \prod_{w \in c} \frac{n_{w^+}}{n_{w^-}} = 1,$$

since $w^+ = w'^-$ for each consecutive edges w, w' of the cycle c .

Conversely, suppose that for a horizontal immersion $g : S \rightarrow M$ the cyclic balance holds. Together with the relation (deg), it allows to find, starting with some copy $S_{v,n}$, $n \in A \setminus \{0\}$, and using the obvious continuation method, a copy of the surface S in the preimage $q^{-1}(g(S)) \subset S \times \mathbb{R}$ distinct from S . This implies the existence of a required covering $p : \widetilde{M} \rightarrow M$ and of an embedding $\tilde{g} : S \rightarrow \widetilde{M}$ with $\tilde{g} \circ p = g$. \square

5.5.2 Compatible symmetric solutions

Here we formulate a proposition which will be used to complete the proof of Theorems 2.3 and 3.1.

A compatible symmetric solution (a, γ) to the BKN-equation over a manifold $M \in \mathfrak{M}$ is said to be a NPC-solution, if it satisfies the condition (NPC)

of Theorem 3.1, i.e., the length function is positive $a > 0$, and the angle function $|\gamma| < 1$.

Proposition 5.19. *If the BKN-equation over $M \in \mathfrak{M}$ has a compatible, symmetric solution (a, γ) , then the same equation has a compatible, symmetric solution (a', γ') which satisfies one of the following two conditions:*

- (1) (a', γ') is a NPC-solution;
- (2) the angle function γ' takes values in $\{0, \pm 1\}$, and $|\gamma'_w| = 1$ for every edge $w \in W$ with $a'_{w-} \cdot a'_{w+} \neq 0$.

Furthermore, if the length function is positive, $a > 0$, then the equation has a compatible, symmetric solution (a', γ') satisfying (1) or

- (3) the length function is positive, $a' > 0$, and the angle function γ' takes values in $\{0, \pm 1\}$.

We prove this Proposition in sect. 5.6.

Remark 5.20. To implement the virtual properties (VE) and (VF) we need rational solutions to the BKN-equation. In the cases (2) and (3), the existence of a rational solution (a', γ') follows from the first part of Proposition 5.10. Therefore, the problem of rationality is reduced to the case of a NPC-solution. There are manifolds $M \in \mathfrak{M}$, for which the BKN-equation has a NPC-solution, but has no rational NPC-solution. However as it is shown in sect. 5.8 (see Proposition 5.26), there always is a characteristic cover of such a manifold for which there is a rational NPC-solution. This suffices for our purposes because we consider virtual properties (see Remark 5.27).

5.5.3 Implementing (VF)

Here we complete the proof of Theorem 3.1(VF). Assume that the BKN-equation over a manifold $M \in \mathfrak{M}$ has a compatible symmetric solution (a, γ) with positive length function, $a > 0$. We show that M is virtually fibered over the circle.

According to Proposition 5.19 and Remark 5.20, one can assume, taking if necessarily an appropriate characteristic covering, that (a, γ) is a rational solution. Then it follows from Proposition 5.2 that there are compatible integer cohomological classes $\{l_v : v \in V\}$ on M such that $l_v(f_v) > 0$ for all vertices $v \in V$ and $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$ (we suppose that fiber orientations of maximal blocks and an orientation of M are fixed). For edges $w \in W$ with $|\gamma_w| < 1$ we have $|l_{-w}(f_w)| < l_w(f_w)$, and for edges with $|\gamma_w| = 1$ we have $l_{-w} = \pm l_w$ due to the symmetry property of γ and compatibility of $\{l_v : v \in V\}$.

Using this collection, we find a horizontal immersion $g : S \rightarrow M$ as in sect. 5.3.4. Namely, for every edge $w \in W$ we put $l_w^+ = l_w + l_{-w}$, $l_w^- = l_w - l_{-w}$.

Then the elements $c_w^+, c_w^- \in H_1(T_{|w|}; \mathbb{Z})$, dual to l_w^+, l_w^- , satisfy the conditions of Lemma 5.6. Note that the numbers $\alpha_w^\pm = c_w^\pm \wedge_w f_w = l_w(f_w) \pm l_{-w}(f_w)$ are positive for edges $w \in W$ with $|\gamma_w| < 1$, and for edges with $|\gamma_w| = 1$ we have $\alpha_w^- \cdot \alpha_w^+ = 0$. Furthermore, $\alpha_w^- \cdot \alpha_w^+ = 0$ if and only if $\alpha_{-w}^- \cdot \alpha_{-w}^+ = 0$ by symmetry of γ .

Since $c_{-w}^+ = c_w^+$, $c_{-w}^- = -c_w^-$, and if $\alpha_w^+ \cdot \alpha_w^- = 0$, then $c_{-w} = \pm c_w$, the horizontal immersions $g_v : S_v \rightarrow M_v$ provided by Lemma 5.6 for every maximal block M_v can be easily pasted together on every gluing torus $T_{|w|}$ at the expense of an appropriate multiplicity d universal for all blocks, and they give a required horizontal immersion $g : S \rightarrow M$.

It follows from the construction of Lemma 5.6 that the graph Γ_g , defined for the immersion g in sect. 5.5.1, has the same vertex set as the graph $\Gamma = \Gamma_M$, $V_g = V$, and its edge set is twice of that of Γ , $W_g = 2W$, in the sense that to every edge $w \in W$ it correspond two edges $w_\pm \in W_g$ with the same end points as w . The number d_{w_\pm} , defined in sect. 5.5.1, is equal to $d \cdot \alpha_w^\pm = d \cdot (l_w(f_w) \pm l_{-w}(f_w))$ in the case $\alpha_w^+ \cdot \alpha_w^- \neq 0$, and $d_{w_-} = d_{w_+} = d \cdot l_w(f_w)$ in the case $\alpha_w^+ \cdot \alpha_w^- = 0$, where the factor $d > 0$ is one and the same for all $w \in W$. It follows from the symmetry condition $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$, $w \in W$, that

$$\frac{d_w}{d_{-w}} = \frac{l_v(f_v)}{l_{v'}(f_{v'})}$$

for all $w \in W_g$, where v is the beginning and v' is the end of w .

This immediately implies the cyclic balance condition for g . Hence, by Lemma 5.18, g is a virtual horizontal embedding, and thus the manifold M virtually fibers over the circle. \square

5.5.4 Implementing (VE) = (NPC) \cup (E)

Here we complete the proof of Theorem 3.1(VE). Assume that the BKN-equation over a manifold $M \in \mathfrak{M}$ has a compatible symmetric solution (a, γ) . We show that M has the property (VE), and (VE) = (NPC) \cup (E). According to Proposition 5.19 one can assume that the solution (a, γ) satisfies one of the conditions (1), (2). In the first case, taking if necessarily an appropriate characteristic covering, we suppose also by Remark 5.20 that the solution is rational. In the second case, one can also suppose by the first part of Proposition 5.10 that the solution is rational (there is no need to take any finite covering in this case, what is important for the proof of the equality (VE) = (NPC) \cup (E)).

In the first case, (a, γ) is a NPC-solution. By sect. 5.5.3, the manifold M virtually fibers over the circle, in particular, it has the property (VE). In the

second case, the solution (a, γ) satisfies the condition (E) of Theorem 3.1. By sect. 5.3.6, the manifold M has the property (E) and hence the property (VE). This also proves the equality $(VE) = (NPC) \cup (E)$. \square

5.5.5 Historical remarks

Ascent Lemma 5.18 is Theorem 2.3 from [RW], where using this result the authors give an example of a horizontal immersion of a closed surface in a graph-manifold which is not a virtual embedding. It was discovered in [Sv1] that the symmetry condition of a solution to the BKN-equation implies the cyclic balance condition for the corresponding horizontal immersion (sect. 5.5.3), and hence the Ascent Lemma can be applied for the proof that a graph-manifold virtually fibers over the circle.

One and the same symmetry condition of solutions to the BKN-equation provides, on the one hand, that the gluings of maximal blocks are isometric while constructing a NPC-metric on a graph-manifold, and, on the other hand, the cyclic balance condition while finding a virtual embedding of a surface in a graph-manifold. This fact looks somewhat mystically.

5.6 Ideals

Here we prove Proposition 5.19 using global methods. The notion of an ideal, the decomposition principle and the polarized discharge Lemma are keys for that. These notions and results play an important role also for the proofs of spectral criteria in sect. 6.

A labeled graph $X = (\Gamma, |B|, K)$ is called an ideal if the BKN-equation over every connected labeled graph X' containing X as a subgraph has a NPC-solution. We describe the most important ideals in the following two lemmas.

Lemma 5.21. *Assume that all charges of a labeled graph X have one and the same sign $\neq 0$. If*

$$(A_X^+ a, a) := \sum_{v \in V} |k_v| a_v^2 - \sum_{w \in W} \frac{a_w - a_{w^+}}{|b_w|} < 0$$

for some nonnegative function $a : V \rightarrow \mathbb{R}$, then X is an ideal.

Lemma 5.22. *Let X be a labeled graph over the circle with odd number of edges $p \geq 1$, or over a linear graph Γ with $p \geq 2$ edges. Assume that in the circle case at most one vertex has a nonzero charge, and if $p = 1$, i.e., in the case of a loop, we require that $|k \cdot b| < 2$, where k is the charge, and b*

the intersection index. In the case of a linear graph, we suppose that only extreme vertices have nonzero charges, which are of the same sign if p is odd, and of different signs, if p is even. Then X is an ideal.

We prove these Lemmas in sect. 5.7. Here, using them, we prove the following sufficient condition for the existence of NPC-solutions. Recall that the operator $H_M : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is defined in sect. 4.1.3.

Lemma 5.23. *Let X be the labeled graph of an oriented manifold $M \in \mathfrak{M}$. If the operator H_M has a negative eigenvalue or the function $s : U \rightarrow \{0, \pm 1\}$ from its definition is zero, then the BKN-equation over X has a NPC-solution.*

Proof. If the sign components graph of X is not bipartite, then the graph X contains one of the ideals described in Lemma 5.22. Thus we suppose that the sign components graph is bipartite.

If the charges $k_v = 0$ for all vertices $v \in V$, then the functions $a \equiv 1$, $\gamma \equiv 0$ form a NPC-solution (a, γ) of the BKN-equation. Thus we also suppose that the function $s : U \rightarrow \{0, \pm 1\}$ takes values in $\{\pm 1\}$.

Since the operator H_M has a negative eigenvalue, there is a sign component $u \in U$, for which the operator $H_u = D_u - J_u$ has a negative eigenvalue. If $u \in U_0$, i.e., if u is a vertex with zero charge, then all edges of the graph Γ_u are loops with common vertex u . In this case, the graph X contains an ideal by Lemma 5.22. Thus we assume that $u \notin U_0$. If $s(u) \text{sign}(u) = -1$, where $\text{sign}(u)$ is the sign of the charges of the vertices from u , then the graph X contains one of the ideals described in Lemma 5.22, because there is a component u_0 with $s(u_0) \text{sign}(u_0) = 1$. The proof is completed in this case. Thus one can assume that $s(u) \text{sign}(u) = 1$. In this case, the operator H_u coincides with operator A_u^+ , and the component u is an ideal by Lemma 5.21. \square

Proof of Proposition 5.19. Consider a quadratic form $F : \mathbb{R}^V \rightarrow \mathbb{R}$, $F(x) = (H_M x, x)$. By Lemma 5.23, one can assume that F is nonnegative on \mathbb{R}^V , and the function $s : U \rightarrow \{0, \pm 1\}$ from the definition of the operator H_M takes values in $\{\pm 1\}$.

On the other hand, for a compatible symmetric solution (a, γ) we have $F(a) \leq 0$ by Lemma 4.3. Hence $F(a) = 0$ and $s(u)\gamma_w = 1$ for all sign components $u \in U$ and all edges $w \in W_u$, for which $a_{w^-} \cdot a_{w^+} \neq 0$. Moreover, the derivative $\partial_v F(a) = 0$ for all $v \in V$. Since

$$\frac{1}{2} \partial_v F(a) = s(u) k_v a_v - \sum_{w \in \partial_0 v} \frac{a_{w^+}}{|b_w|},$$

where $\partial_0 v = \partial v \cap W_u$, it follows from vanishing of the derivative and from that the length function a is nonnegative that if $a_v = 0$ for some vertex $v \in u$,

then the function a vanishes at all vertices of the component u . Since $a \not\equiv 0$, there is a component $u \in U$, over which the function a is positive. Using the BKN-equation for (a, γ) we obtain from vanishing of the derivative that

$$\sum_{w \in \partial v \setminus \partial_0 v} \frac{\gamma_w}{|b_w|} a_{w^+} = 0 \quad (\text{G})$$

for all $v \in u$.

Define functions $a' : V \rightarrow \mathbb{R}$, $\gamma' : W \rightarrow \{0, \pm 1\}$ by $a'_v = a_v$ for all $v \in u$ and $a'_v = 0$ for all $v \notin u$; $\gamma'_w = \gamma_w$ for all $w \in W_u$ and $\gamma'_w = 0$ for all $w \notin W_u$. It follows from the equality (G) that (a', γ') is a compatible symmetric solution to the BKN-equation over X , and from the preceding we obtain that $|\gamma'_w| = 1$ for all edges $w \in W$ with $a'_{w^-} \cdot a'_{w^+} \neq 0$. This is the case (2).

Assume now that $a_v > 0$ for all vertices $v \in V$. We put $a' = a$; $\gamma'_w = \gamma_w$ for all components $u \in U$ and all edges $w \in W_u$ and $\gamma'_w = 0$ for all edges w connecting vertices from different sign components. Then as above (a', γ') is a compatible symmetric solution to the BKN-equation over X with positive length function, $a' > 0$, and angle function γ' taking the values in $\{0, \pm 1\}$. This is the case (3). \square

5.7 Decomposition principle

Here we prove Lemmas 5.21 and 5.22. Our main tool is the decomposition of labeled graphs $X = (\Gamma, |B|, K)$ into dipoles.

5.7.1 Dipole

The graph with two vertices v, v' and one nonoriented edge $(w, -w)$ between them, $w \in \partial v$, will be called *the dipole*. The BKN-equation over a labeled dipole $D = (k_w, k_{-w}; b_w)$ is

$$k_w a_w = \frac{\gamma_w}{|b_w|} a_{-w}, \quad k_{-w} a_{-w} = \frac{\gamma_{-w}}{|b_w|} a_w,$$

where k_w, k_{-w} are the charges of the vertices v, v' respectively, $b_w \in \mathbb{Z} \setminus \{0\}$ is the intersection index. We note that labeled dipoles (as well as other labeled graphs) considered here are not necessarily associated with graph-manifolds.

All NPC-solutions $(a_w, a_{-w}; \gamma_w)$ to this equation can be described as follows. The positive numbers a_w, a_{-w} are defined up to a common positive factor. It follows from the condition $a_w \cdot a_{-w} \neq 0$ that $\gamma_w^2 = k_w k_{-w} b_w^2$. This means, in particular, that for different signs of the charges, $k_w k_{-w} < 0$, or

for $k_w k_{-w} b_w^2 \geq 1$ the BKN-equation of a dipole has no NPC-solution. Thus we assume that $0 \leq k_w k_{-w} b_w^2 < 1$.

For $k_w k_{-w} > 0$ the sign of γ_w coincides with the sign of the charges k_w , k_{-w} and

$$\left(\frac{a_w}{a_{-w}} \right)^2 = \frac{k_{-w}}{k_w}. \quad (\text{dip})$$

In the case $k_w = k_{-w} = 0$, the equation put no restriction to (a_w, a_{-w}) . When one of the charges vanishes and another not, there obviously is no NPC-solution.

5.7.2 Conjunction and decomposition of labelled graphs

Under conjunction operation of labeled graphs X' , X'' (the case $X' = X''$ is not excluded) some of their vertices are glued and their charges are added (notation: $X = X' + X''$). This operation corresponds to toral summation of oriented graph-manifolds along some maximal blocks (for details see [BK1]). The converse operation is called a decomposition of a labeled graph.

The decomposition principle claims roughly speaking that compatible solutions to the BKN-equation are parameterized (with some reservation) by decompositions of the labeled graph into labeled dipoles satisfying appropriate conditions.

Here is the precise statement. The decomposition principle says that

(a) if (a', γ') , (a'', γ'') are compatible solutions to the BKN-equations over labeled graphs X' , X'' respectively, $X = X' + X''$ is the conjunction along some vertices, and the values of the functions a' , a'' at the glued vertices coincide (in the case $X' = X''$ we require $(a', \gamma') = (a'', \gamma'')$), then (a, γ) is a compatible solution to the BKN-equation over the labeled graph X , where the functions a , γ are naturally defined by the solutions (a', γ') , (a'', γ'') : their values coincide with the values of corresponding functions of that solution which is defined along the considered vertices or edges.

(b) Conversely, any compatible solution (a, γ) to the BKN-equation over a labeled graph X with positive length function a canonically defines a decomposition of the graph $X = \sum_{e \in E} D_e$ into labeled dipoles $D_e = (k_w, k_{-w}; b_w)$, where $e = (w, -w)$ is a nonoriented edge, according to the conditions: their intersection indices b_w are the same as of the graph X ; for every edge $w \in W$ the charge of the vertex $v = w^-$ of the decomposition dipole $D_{(w, -w)}$ is defined by $k_w = \frac{\gamma_w}{|b_w|} \cdot \frac{a_{w+}}{a_{w-}}$. Furthermore, the solution (a, γ) restricted to any decomposition dipole is a compatible solution to the BKN-equation for that dipole.

Here we use the notation of an edge w as the subscript in the notation

of the vertex charge of a dipole because this does not lead to any ambiguity and distinguishes the obtained charge from the corresponding charge of X . Moreover, this is convenient for writing the relation $k_v = \sum_{w \in \partial v} k_w$, $v \in V$, between the charge k_v of X and the charges k_w of the dipoles into which X is decomposed by a compatible solution.

5.7.3 The vertex balance

The mentioned above equality $k_v = \sum_{w \in \partial v} k_w$, $v \in V$, we call the equation of the vertex balance. It is a necessary condition for the decomposition of a labeled graph into dipoles defined by a compatible solution to the BKN-equation. Moreover, the vertex balance equation together with the BKN-equations over the dipoles of a decomposition $X = \sum_{e \in E} D_e$ for collections $(a_w, a_{-w}; \gamma_w, \gamma_{-w})$, $e = (w, -w)$, implies the BKN-equation over X for (a, γ) , if it is possible to adjust the collections (a_w, a_{-w}) by positive factors so that their elements coincide at the glued vertices and hence define a required length function a . In particular, if the graph Γ of a labeled graph $X = (\Gamma, |B|, K)$ is a tree, then any decomposition $X = \sum_{e \in E} D_e$ into labeled dipoles, each of which $D_e = (k_w, k_{-w}; b_w)$, $e = (w, -w)$, has a compatible symmetric solution $(a_w, a_{-w}; \gamma_w)$ to its BKN-equation with positive a_w, a_{-w} , defines a compatible symmetric solution (a, γ) to the BKN-equation over X . Namely, $\gamma(w) = \gamma_w$ and $a_v/a_{v'} = a_w/a_{-w}$ for every edge $w \in W$, where $v = w^-$ is the beginning of w , and $v' = w^+$ is its end. Since there is no nontrivial circuit, it is possible to define inductively a function $a : V \rightarrow \mathbb{R}$ using the relations $a_v/a_{v'} = a_w/a_{-w}$, the restriction of which to every dipole D_e differs from (a_w, a_{-w}) , $e = (w, -w)$, possibly by a common positive factor only. Then (a, γ) is a required solution to the BKN-equation according (a) of the decomposition principle.

5.7.4 Polarized discharge of a vertex

Lemma 5.24. *Let $X = (\Gamma, |B|, K)$ be a labeled graph with simply connected graph Γ . Given its vertex $v \in V$, there are $\delta \in \{0, \pm 1\}$ and $\varepsilon > 0$ such that if $|k'_v| < \varepsilon$, $\text{sign}(k'_v) = \delta$, and the charges of the remaining vertices are the same as of X , $k'_{v'} = k_{v'}$, then the BKN-equation over the labeled graph $X' = (\Gamma, |B|, K')$ has a NPC-solution.*

Proof. We argue by induction over the number of vertices. For a dipole the assertion immediately follows from the description of NPC-solutions to its BKN-equation. In general case, if v is an interior vertex of the tree Γ , then X can be represented as the conjunction along v of a finite collection of trees each of which has less vertices than X . Then the assertion follows from

the inductive assumption and (a) of the decomposition principle because the graph Γ is a tree.

Now, assume that for the vertex v there is a unique neighboring vertex v' in Γ . Then X can be represented as the conjunction along v' of a dipole D with vertices v, v' and a labeled tree X_1 . By the inductive assumption, one can suppose that the charge $k_{v'}^1$ of the vertex v' of X_1 is close to zero and has a sign so that the BKN-equation over X_1 has a NPC-solution. Then the dipole charge $k_{v'}^D$ is defined from the vertex balance $k_{v'}^D + k_{v'}^1 = k_{v'}$. However, whichever is the charge $k_{v'}^D$, it is always possible to find a sign $\delta \in \{0, \pm 1\}$ so that if the dipole charge k_v^D at v has the sign δ and is sufficiently close to zero, then the BKN-equation over D has a NPC-solution. As above, the assertion now follows from the decomposition principle. \square

For the proof of Lemmas 5.21 and 5.22 we shall use the following characteristic of ideals.

Lemma 5.25. *Suppose that a labeled graph $X = (\Gamma, |B|, K)$ has the following property: for every charge collection $K' \in \mathbb{R}^V$ from some neighborhood in \mathbb{R}^V of K , the BKN-equation over $X' = (\Gamma, |B|, K')$ has a NPC-solution. Then X is an ideal.*

Proof. Assume that X is the subgraph of a connected labeled graph X_0 . We describe how to find a NPC-solution to the BKN-equation over X_0 . The graph X_0 can be represented as the conjunction of X' , a disjoint collection of trees Y and a collection of dipoles Z with zero charges,

$$X_0 = X' + Y + Z.$$

Furthermore, $X' = (\Gamma, |B|, K')$, and every tree Y_α from Y has a charge of the conjunction vertex which is sufficiently close to zero and satisfies the condition of Lemma 5.24. It follows from the vertex balance that the charge collection K' can be chosen arbitrarily close to the initial collection $K \in \mathbb{R}^V$. Since the dipoles with zero charges put no restriction to the length function of a solution to the BKN-equation, we complete the proof applying the decomposition principle and taking into account that the collection Y consists of trees. \square

Proof of Lemma 5.21. To be definite, we assume that all charges of the labeled graph X are positive. Arguing by induction over the number of vertices, we assume that the assertion is true for all labeled graph with number of the vertices less than that of $X = (\Gamma, |B|, K)$. By the condition, the quadratic form A_X^+ on \mathbb{R}^V has a negative eigenvalue. There is a sufficiently small neighborhood in \mathbb{R}^V of K such that for every K' from this neighborhood the form $A_{X'}^+$, $X' = (\Gamma, |B|, K')$, has a negative eigenvalue.

Consider the family of forms $A_{X',t}^+$, $0 \leq t \leq 1$,

$$(A_{X',t}^+ x, x) := \sum_{v \in V} k'_v x_v^2 - \sum_{w \in W} \frac{t}{|b_w|} x_w - x_{w+}.$$

For $t = 0$ the corresponding form of this family is positive definite, and for $t = 1$ it has a negative eigenvalue. Thus there is $t_0 \in (0, 1)$ such that the form A_{X',t_0}^+ has a zero eigenvalue. Let $a_0 \in \mathbb{R}^V$ be a corresponding eigenvector. Changing the sign of $\gamma_w^0 = t_0$ for the edges w leaving the vertices at which a_0 is negative and simultaneously changing the sign of the negative values of a_0 , one can assume that $a_0 \geq 0$. Then (a_0, γ_0) is a symmetric solution to the BKN-equation over X' , for which the length function a_0 is nonnegative, and the angle function $|\gamma_0| < 1$. One can also assume that this solution is compatible.

Consider the labeled subgraph $X_0 \subset X'$ spanned by the vertices at which a_0 is positive. The restriction of (a_0, γ_0) to this subgraph is a NPC-solution to the BKN-equation over X_0 (the subgraph has, obviously, no isolated vertex, i.e., a vertex without incident edges). It follows that $(A_{X_0}^+ a_0, a_0) < 0$ for the form $A_{X_0}^+$. Thus this form has a negative eigenvalue.

If X_0 is a proper subgraph in X' , then by the inductive assumption it is an ideal. Thus in any case, $X_0 = X'$ or $X_0 \neq X'$, the BKN-equation over X' has a NPC-solution. By Lemma 5.25, the labeled graph X is an ideal. \square

Proof of Lemma 5.22. In the case of a linear Γ , every labeled graph $X' = (\Gamma, |B|, K')$ can be represented as the conjunction of p labeled dipoles $D'_e = (k'_w, k'_{-w}; |b_w|)$, where $e = (w, -w)$ is a nonoriented edge, satisfying the NPC-condition $0 < \gamma_w^2 := k'_w k'_{-w} b_w^2 < 1$ (see sect. 5.7), if the charge collection $K' \in \mathbb{R}^V$ is sufficiently close to K . By the decomposition principle, the BKN-equation over X' has a NPC-solution because the graph Γ is simply connected. Now, it follows from Lemma 5.25 that X is an ideal.

In the loop case ($p = 1$), the BKN-equation over X has a NPC-solution due to the condition $|k \cdot b| < 2$. This condition is open in the charge space, thus X is an ideal.

Now, we assume that Γ is a circle with odd number $p \geq 3$ of edges. If every charge of X is zero, then the functions $a \equiv 1$, $\gamma \equiv 0$ make up a NPC-solution (a, γ) to the BKN-equation. Let X' be a perturbed graph. One can assume that there is a vertex $v_1 \in V$, with nonzero charge k'_1 . We assume further that the value $|k'_1|$ is maximal over all vertices. Decompose X' into dipoles $D'_e = (k'_w, k'_{-w}; |b_w|)$ with all nonzero charges so that together with NPC-condition above the following conditions are fulfilled: $k'_w = k'_{-w}$ for all edges $e = (w, -w)$ not incident to v_1 . For the edges $e_1 = (w_1, -w_1)$,

$e_p = (w_p, -w_p)$, incident to v_1 , $v_1 = w_1^- = w_p^+$, we require that

$$\frac{k'_{w_1}}{k'_{-w_1}} = \frac{k'_{-w_p}}{k'_{w_p}}.$$

In view of the relation (dip) from sect. 5.7, these conditions provide the existence of a NPC-solution to the BKN-equation over X' . It is easy to see that if K' is sufficiently close to K , then a required decomposition of the labeled circle X' into dipoles does exist because $p \geq 3$ is odd. Hence, X' is an ideal. \square

5.7.5 Historical remarks

The notion of ideals, the decomposition principle and the lemma about a polarized discharge of a vertex have appeared in [BK1], Lemma 5.25 is taken from [BK2].

5.8 Symmetric rational solutions

Here, under some restrictions, we find a rational approximation of NPC-solutions to the BKN-equation.

Proposition 5.26. *Let $X = (\Gamma, |B|, K)$ be the labeled graph of a manifold $M \in \mathfrak{M}$ such that if the graph Γ is bipartite, then there are at least four vertices lying in one and the same part with equal nonzero charges. Then every NPC-solution (a, γ) to the BKN-equation over X can be approximated by rational NPC-solutions.*

Remark 5.27. There are graph-manifolds $M \in \mathfrak{M}$, for which the BKN-equation has a real NPC-solution (a, γ) , and has no rational NPC-solution. As such an M one can take an “irrational dipole” whose graph consists of two vertices v, v' and one (nonoriented) edge e between them, the charges and the intersection index satisfy the condition $0 < k_v k_{v'} b_e^2 < 1$ and $\sqrt{k_v k_{v'}} \notin \mathbb{Q}$ (at least one of the blocks of M with these properties must have singular fibers).

On the other hand, if a manifold M has a nonzero charge, then it is easy to find its characteristic covering $\widetilde{M} \rightarrow M$ such that the (connected) \widetilde{M} satisfies the conditions of Proposition 5.26. In the case when all the charges are equals zero, the functions $a \equiv 1$ and $\gamma \equiv 0$ form a NPC-solution (a, γ) to the BKN-equation as it was already mentioned in the proof of Lemma 5.23. Therefore, while considering the virtual properties (VE), (VF), one can assume that NPC-solutions are rational by taking an appropriate characteristic covering if necessarily.

The proof of Proposition 5.26 is based on properties of the incidence matrix of the graph Γ , which is defined as the map $I = I(\Gamma) : V \times E \rightarrow \{0, 1, 2\}$ (E is the set of nonoriented edges of Γ), $I_{ve} \neq 0$ if and only if the vertex $v \in V$ and the edge $e \in E$ are incident, furthermore, $I_{ve} = 2$ for the loops e with the vertex v and only for them. The incidence matrix can also be identified with a linear map $I : \mathbb{R}^E \rightarrow \mathbb{R}^V$ by considering every restriction $I_v = I|_{v \times E}$ as the row corresponding to the vertex $v \in V$.

Lemma 5.28. *If a connected graph Γ is not bipartite, then the rank of the incidence matrix I is equal to the number of the vertices, $|V|$. If the graph Γ is bipartite, then the rank of I is equal to $|V| - 1$.*

Proof. Assume that there is a zero linear combination of rows, $\sum_{v \in V} c_v I_v = 0$. Then for every edge $e \in E$, we have $c_v + c_{v'} = 0$, where v, v' are the ends of e (possibly coinciding), because the column of I corresponding to e has nonzero elements for the vertices v and v' only. It follows that if the graph Γ has a circuit with odd number of edges, i.e., it is not bipartite, then $c_v = 0$ for the vertices of that circuit, and hence for all vertices $v \in V$ by connectedness of Γ . In this case, the rank of I is equal to $|V|$.

Assume now that the graph Γ is bipartite, and let $V = V_0 \cup V_1$ be a decomposition of the vertex set into (nonempty) parts. Then

$$\sum_{v \in V_0} I_v - \sum_{v \in V_1} I_v = 0$$

is the unique (up to a nonzero factor) nontrivial combination of rows of I , and hence its rank is equal to $|V| - 1$. \square

There are several interpretations of the BKN-equation. The most convenient one for symmetric rational approximations is the following. For a positive length function a , the BKN-equation over X can be written in more symmetric equivalent form

$$k_v a_v^2 = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} a_{w^-}, \quad v \in V.$$

If in addition the angle function γ is symmetric, then this form in turn is equivalent to the equation

$$A = I(D), \tag{+}$$

where $I : \mathbb{R}^E \rightarrow \mathbb{R}^V$ is the incidence matrix, $A \in \mathbb{R}^V$, $A_v = k_v a_v^2$, and $D \in \mathbb{R}^E$, $D_e = \frac{\gamma_w}{|b_w|} a_{w^+} a_{w^-}$ for every edge $e = (w, -w)$.

Proof of Proposition 5.26: the graph Γ is not bipartite. In this case, we have $|V| \leq |E|$, because the graph Γ is not simply connected. Since the rank of the matrix I equals $|V|$, there is a linearly independent collection of its rows $E_0 \subset E$, $|E_0| = |V|$, and by a classical theorem from linear algebra, the solutions to the symmetrized BKN-equation (+) can be parameterized by $|V|$ linear functions L_e , $e \in E_0$ with rational coefficients depending on $|E| = |V| + (|E| - |V|)$ arguments A_v , $v \in V$, $D_{e'}$, $e' \in E \setminus E_0$,

$$\frac{\gamma_w}{|b_w|} a_{w^+} a_{w^-} = L_e(A_v, D_{e'}), \quad e = (w, -w) \in E_0.$$

Approximating the numbers a_v , $v \in V$, and $\gamma_{w'}$, $(w', -w') \in E \setminus E_0$ by rational numbers a'_v and $\gamma'_{w'}$ respectively, we obtain a rational approximation

$$\gamma'_w = \frac{|b_w|}{a'_{w^+} a'_{w^-}} L_e(A'_v, D'_{e'})$$

of γ_w for all $(w, -w) \in E_0$. This gives a required approximation of the solution (a, γ) to the BKN-equation by compatible, symmetric, rational solutions. \square

Now, we assume that the graph Γ is bipartite, and let $V = V_0 \cup V_1$ be a decomposition of its vertex sets into parts, $U \subset V_0$ be a subset consisting of four vertices with equal charges, $k_v = k \neq 0$ for all $v \in U$. Consider the set $Q = \{x \in \mathbb{R}^V : \sum_{v \in V_0} k_v x_v^2 = \sum_{v \in V_1} k_v x_v^2\}$. The equality $\sum_{v \in V_0} I_v = \sum_{v \in V_1} I_v$ implies that for any solution (a, γ) to the BKN-equation (+), the length function a lies in Q , in particular, the set Q contains points with all coordinates different from zero.

We use the condition $|U| = 4$ to apply a classical result of Lagrange that every natural number can be represented as the sum of four squares of integer numbers for the proof of the following Lemma.

Lemma 5.29. *The points with rational coordinates are dense in Q .*

Proof. W.L.G. we assume that $k_v \neq 0$ for all $v \in V$. Put for brevity $\kappa_v := -k_v$ for $v \in V_0$ and $\kappa_v := k_v$ for $v \in V_1$. Then $Q = \{x \in \mathbb{R}^V : \sum_{v \in V} \kappa_v x_v^2 = 0\}$.

The Lagrange theorem implies that the set Q contains a point $c \neq 0$ with rational coordinates. Indeed, approximating a point $a \in Q$ with $a_v \neq 0$ for all $v \in V$ by rational points from \mathbb{R}^V , we find a rational $q > 0$, which can be represented as

$$q = \frac{1}{k} \sum_{v \in V \setminus U} \kappa_v c_v^2,$$

where $c_v \in \mathbb{Q}$ for all $v \in V \setminus U$. Representing q as the sum of four squares of rationals, $q = \sum_{v \in U} c_v^2$, by the Lagrange theorem, we get a required point $c \in Q$.

It suffices to show that the rational points are dense in the subset $Q_0 \subset Q$ of the points which have all coordinates different from zero. Take $b \in Q_0$ assuming that $c \neq b$. Moreover, changing if necessarily the signs of coordinates of c , one can assume that $A := \sum_{v \in V} \kappa_v b_v c_v > 0$.

Consider the line σ in \mathbb{R}^V through the points c and b , $\sigma(t) = c + t\xi$, where $\xi = b - c$, $t \in \mathbb{R}$. The set Q is the zero set of the function $F : \mathbb{R}^V \rightarrow \mathbb{R}$, $F(x) = \sum_{v \in V} \kappa_v x_v^2$. The gradient of F is different from zero at any point $x \in Q \setminus \{0\}$, thus $Q \setminus \{0\}$ is a regular hypersurface in \mathbb{R}^V . The restriction of F to the line σ is a quadratic polynomial, $F(t) = F \circ \sigma(t)$, which vanishes at $t = 0, 1$. Thus $F(t) = A't(t - 1)$ for some coefficient $A' \in \mathbb{R}$. A direct computation shows that $A' = -2A \neq 0$. Consequently, the line σ intersects transversally Q at b . Now, approximating the vector ξ by vectors $\xi' \in \mathbb{R}^V$ with rational coordinates, we obtain lines σ' , $\sigma'(t) = c + t\xi'$, whose intersection points with Q gives a rational approximation of b . \square

Proof of Proposition 5.26: Γ is bipartite. The argument for the nonbipartite case can be applied also here with the following modifications: a linearly independent collection of rows $E_0 \subset E$ consists of $|V| - 1$ elements, and the length function is chosen from the set $Q \subset \mathbb{R}^V$, $a \in Q$. Lemma 5.29 provides now a required rational approximation. \square

6 Proof of spectral criteria

6.1 Spectral criterion for (I)=(HI)

Here we prove Theorem 4.4.

Only if. Assume that the BKN-equation over M has a compatible solution (a, γ) , and we can also assume that the length function is positive, $a > 0$. By Lemma 4.1, the operator A_M^+ has a nonpositive eigenvalue, and if it has no negative eigenvalue, then the length function $a \in \mathbb{R}^V$ lies in the kernel of this operator and $\gamma_w = \text{sign}(k_w -)$ for every edge $w \in W$. Since $\gamma_w \cdot \gamma_{-w} \neq -1$ by compatibility of the solution, all charges of M have one and the same sign. \square

If. First, consider the case when the operator A_M^+ has no negative eigenvalue, and all charges of M have one and the same sign. One can assume that all charges are positive. By the condition, the operator A_M^+ is degenerate. Let

$a \in \mathbb{R}^V$ be a nonzero vector from its kernel. Then the functions $|a|$ and $\gamma_w = \text{sign}(a_{w-}) \cdot \text{sign}(a_{w+})$, $w \in W$, are the length function and the angle function respectively of a compatible solution to the BKN-equation over M . By Theorem 3.1, the manifold M has the property (I)=(HI).

Now, we assume that the operator A_M^+ has a negative eigenvalue. Then the argument similar to the proof of Lemma 5.21 shows that the deformation $A_{M,t}^+$, $0 \leq t \leq 1$, of A_M^+ has for some $t \in [0, 1)$ a zero eigenvalue, which easily give a compatible solution to the BKN-equation. Thus also in this case M has the property (I)=(HI). \square

6.2 Spectral criterion for (E)

Here we prove Theorem 4.5.

Only if. Assume a manifold $M \in \mathfrak{M}$ has the property (E). According Theorem 3.1(E), there is a compatible, symmetric solution (a, γ) to the BKN-equation over M such that if $a_{w-} \cdot a_{w+} \neq 0$ then $\gamma_w = \pm 1$ for every edge $w \in W$. Define a cocycle $\lambda : W \rightarrow \mathbb{Z}_2$ putting $\lambda_w = \gamma_w$, if $a_{w-} \cdot a_{w+} \neq 0$, and $\lambda_w = 1$ otherwise. Let us show that the operator A_λ is weakly degenerate.

Suppose $a_v \neq 0$ for some $v \in V$. It suffices to show that $(A_\lambda a)_v = 0$. We have

$$(A_\lambda a)_v = k_v a_v - \sum_{w \in \partial v} \frac{\lambda_w}{|b_w|} a_{w+} = k_v a_v - \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w+} = 0$$

by the BKN-equation, since $\lambda_w a_{w+} = \gamma_w a_{w+}$ for $w \in \partial v$ whatever a_{w+} is zero or not. \square

If. Assume that the operator A_λ is weakly degenerate for some class $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$. Take a nonzero vector $x \in \mathbb{R}^V$ with property $(A_\lambda x)_v = 0$ for all vertices $v \in V$ with $x_v \neq 0$. Define a function $\gamma : W \rightarrow \{0, \pm 1\}$ by $\gamma_w = \lambda_w \text{sign}(x_{w-} \cdot x_{w+})$, $w \in W$. Then $(|x|, \gamma)$ is a compatible, symmetric solution to the BKN-equation over M satisfying the condition (E) of Theorem 3.1. Thus M has the property (E). \square

6.3 Spectral criterion for (VE)

Here we prove Theorem 4.6.

Only if. Assume a manifold $M \in \mathfrak{M}$ has the property (VE). According Theorem 3.1 there is a compatible, symmetric solution (a, γ) to the BKN-equation over M . Then by Lemma 4.3 the operator H_M has a nonpositive eigenvalue. \square

If. Assume the operator H_M has a nonpositive eigenvalue. According Theorem 3.1(VE), it suffices to show that the BKN-equation over M has a compatible, symmetric solution. If H_M has a negative eigenvalue or the function s from its definition is zero, then this follows from Lemma 5.23. Thus we assume that the operator H_M is degenerate and the function s takes the values in $\{\pm 1\}$. Let $a \in \mathbb{R}^V$ be a nonzero vector from the kernel of H_M . Then

$$s(u)k_v a_v - \sum_{w \in \partial v \cap W_u} \frac{a_{w^+}}{|b_w|} = 0$$

for all sign components $u \in U$ and all vertices $v \in V_u$. Define $\gamma_w = s(u) \text{sign}(a_{w^-} \cdot a_{w^+})$ for $w \in W_u$ and $\gamma_w = 0$ for every edge $w \in W$ connecting vertices from different sign components. Then $(|a|, \gamma)$ is a compatible, symmetric solution to the BKN-equation over M . \square

6.4 Spectral criterion for (F)

Here we prove Theorem 4.7.

Only if. By Theorem 3.1(F), there is a function $\varepsilon : V \rightarrow \{\pm 1\}$ and a compatible solution (a, γ) to the BKN-equation such that $a_v > 0$ for all $v \in V$ and $\gamma_w = \varepsilon_{w^-} \varepsilon_{w^+} \text{sign}(b_w)$ for all $w \in W$. It suffices to show that the vector $x = (x_v) \in \mathbb{R}^V$, defined by $x_v = \varepsilon_v a_v$, lies in the kernel of the operator A_ρ . By Lemma 4.2 one can assume that

$$J_M^\rho x = \sum_{v \in V} \left(\sum_{w \in \partial v} \frac{1}{b_w} x_{w^+} \right) v.$$

Thus

$$A_\rho x = \sum_{v \in V} \left(k_v x_v - \sum_{w \in \partial v} \frac{1}{b_w} x_{w^+} \right) v = \sum_{v \in V} \varepsilon_v \left(k_v a_v - \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} \right) v = 0.$$

\square

If. Assume the operator $A_\rho : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is supersingular. Let $a \in \mathbb{R}^V$ be an element from its kernel with all nonzero coordinates, $a_v \neq 0$, $v \in V$. Then for the function $\varepsilon : V \rightarrow \{\pm 1\}$, $\varepsilon_v = \text{sign}(a_v)$, the functions $|a| : V \rightarrow \mathbb{R}$ and $\gamma : W \rightarrow \{\pm 1\}$, $\gamma_w = \varepsilon_{w^+} \varepsilon_{w^-} \text{sign}(b_w)$, are the length function and the angle function respectively of a solution to the BKN-equation, which satisfies the condition (F) of Theorem 3.1. Thus the manifold M fibers over the circle. \square

6.5 Spectral criterion for (VF)

Here we prove Theorem 4.8.

Only if. Assume that a manifold $M \in \mathfrak{M}$ is virtually fibered over the circle. Then by Theorem 3.1 there is a compatible, symmetric solution (a, γ) to the BKN-equation over M with positive length function, $a > 0$. Assume the operator H_M has no negative eigenvalue. Then $(H_M a, a) \geq 0$. On the other hand $(H_M a, a) \leq 0$ by Lemma 4.3. Thus $(H_M a, a) = 0$. Since H_M is positive semidefinite, the vector a lies in its kernel and hence H_M is supersingular. \square

If. By Theorem 3.1(VF) it suffices to show that the BKN-equation over M has a compatible, symmetric solution with positive length function. If the operator H_M has a negative eigenvalue or the function s from its definition equals zero, then this follows from Lemma 5.23.

One can assume that the operator H_M is positive semidefinite and supersingular, and the function s takes the values in $\{\pm 1\}$. Take $a \in \mathbb{R}^V$ from the kernel of H_M such that $a_v \neq 0$ for all $v \in V$. Define a function $\gamma : W \rightarrow \{0, \pm 1\}$ putting $\gamma_w = 0$ for all edges $w \in W$ connecting vertices from different sign components, and $\gamma_w = s(u) \text{sign}(a_{w-} \cdot a_{w+})$ for all sign components $u \in U$ and all $w \in W_u$. Then $(|a|, \gamma)$ is a compatible, symmetric solution to the BKN-equation over M with positive length function. \square

6.6 Spectral criterion for (NPC)

Here we prove Theorem 4.9.

Only if. Assume $M \in \mathfrak{M}$ carries a NPC-metric. Then by Theorem 3.1(NPC) there is a compatible, symmetric solution (a, γ) to the BKN-equation over M with positive length function, $a > 0$, and the angle function $|\gamma| < 1$.

Assume the operator H_M has no negative eigenvalue. Thus $(H_M a, a) \geq 0$. On the other hand $(H_M a, a) \leq 0$ by Lemma 4.3. Hence $(H_M a, a) = 0$ and moreover $s(u)\gamma_w = 1$ for every sign component $u \in U$ and every edge $w \in W_u$. Together with condition $|\gamma| < 1$ this means that every sign component consists of one point, and $W_u = \emptyset$. In this case $(H_M x, x) = \sum_{v \in V} s(v)k_v x_v^2$ for every $x \in \mathbb{R}^V$.

Assume the function s takes the values in $\{\pm 1\}$. Then $s(v_0) = 1$ for some vertex $v_0 \in V$ with positive charge, $k_{v_0} > 0$. Since $0 = (H_M a, a) = \sum_{v \in V} s(v)k_v a_v^2$, this means that the operator H_M has a negative eigenvalue, a contradiction. Thus $s \equiv 0$. \square

If. By Theorem 3.1(NPC) it suffices to prove that the BKN-equation over M has a NPC-solution. This is proved in Lemma 5.23. \square

6.6.1 Historical remarks

As we already mentioned, the spectral criterion for (NPC) obtained in [BK2] is inaccurate. Namely, in terms of this survey, it claims that the property (NPC) is equivalent to that the operator H_M has a negative eigenvalue, or $H_M \equiv 0$. As a counterexample one can take a graph-manifold $M \in \mathfrak{M}$, which is the mapping torus of a Dehn twist of an orientable surface of the genus ≥ 2 along a simple closed curve not homological to zero. The graph of such a manifold is a loop, the charge of the vertex equals $k = 2/|b|$, where the intersection index b coincides with the order of the twist, and $H_M \equiv 0$. However, M admits no NPC-metric because $s \not\equiv 0$. Nevertheless, all results of [BK2], which use the mentioned criterion, remain true, in particular, this holds for the criterion for mapping tori of Dehn twists to have a NPC-metric (Theorem 4.6.3).

A similar inaccuracy has occurred in the criterion for the property of an infinite graph-manifold to carry a NPC-metric with bounded sectional curvatures and finite volume, obtained in [BK3]. The error is also not a principal one, and it can be corrected similarly (in term of this survey) by replacing the condition $H_M \equiv 0$ by the condition $s \equiv 0$.

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